Final Exam: Analysis-I (202200143), MOD-01-AM: Structures and Models

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Date/Time: 15-December-2022, 08:45 – 11:45

- Closed book exam! Use of own text-material or an electronic calculator is not allowed.
- All answers must be motivated, including the answers of Section C.
- Answers for Section A *must* use the four steps (practised during Tutor Sessions).
  - (i.) Get Started: describe what the problem is about and your initial thoughts
  - (ii.) Devise Plan: provide an outline how you plan to solve (or have solved) the problem
  - (iii.) Execute: execute your plan (and try) to reach your solution
  - (iv.) Evaluate: reflect on your solution and/or approach

  Points are distributed (roughly) as: steps (i.)+(ii.) 35%, step (iii.) 50% and step (iv.) 15%.
- Section Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: $0.7 \times \text{Grade\_Section\_A} + 0.3 \times \text{Grade\_Section\_C}$ (rounded off according to EER)
- Good Luck!

Section C:

Total Points : 15

1. (a) Let $z \in \mathbb{C}$ with $|z| = 1$ and $\mathrm{Re}(z^2) = -1/2$. Compute $\left|z^2 + z^4\right|$. [4]

   (b) Solve for all the roots of the following equation: [3]

$$z^4 = 8\sqrt{2}(1+i)e^{3i\pi/4} \qquad (z \in \mathbb{C}).$$

   Express them in the form $a + ib$, where $a, b \in \mathbb{R}$.

2. Let the function $f : \mathbb{R} \setminus \{-2, 2\} \to \mathbb{R}$ be given by

$$f(x) = \frac{x-1}{x^2-4}, \qquad x \in \mathbb{R} \setminus \{-2, 2\}.$$

   Show, by mathematical induction, that the nth-order derivative of f is given by [8]

$$f^{(n)}(x) = \frac{n!(-1)^n}{4}\left(\frac{3}{(x+2)^{n+1}} + \frac{1}{(x-2)^{n+1}}\right), \quad n \in \mathbb{N}, \quad x \in \mathbb{R} \setminus \{-2, 2\}.$$

   [Hint: You may want to start by showing that $f(x) = \frac{3}{4} \cdot \frac{1}{x+2} + \frac{1}{4} \cdot \frac{1}{x-2}$.]

[Section A is on page 2.]

Section A: [Follow the four-step procedure.] Total Points : 35

3. Answer the following. [You may may give one (combined) "Evaluation".] [4+6+2]

   (a) Let the sequence $\{x_n\}$ be given by

   $$x_n = \frac{2n^2 + 2n + 1}{3n^2 + 1}, \qquad n \in \mathbb{N}.$$

   Prove, using the definition, that the sequence converges.

   (b) Using different laws of limit, compute the following:

   $$\lim_{n\to\infty} \left[\sqrt{4n^2 + n} - 2n + \frac{e^{-n}}{n} \right].$$

   [Hint: Splitting into different suitable sequences may help.]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be cotinuous at $a \in \mathbb{R}$. Prove that if $f(a) < M$ for some $M \in \mathbb{R}$, then there is an open interval I containing a such that $f(x) < M$ for all $x \in I$. [4+5+1]

5. (a) Let $a < b$ be two real numbers. Suppose that $f : (a, b) \to \mathbb{R}$ is a differentiable function such that $f'(x) < 0$ for all $x \in (a, b)$.
   Prove that f is strictly decreasing on (a, b). [2+2+1]

   (b) Show that [3+4+1]

   $$x^2 e^{-x} < 0.8 - e^{-x} \quad \text{for all} \quad x > 1.$$

   [Recall that $e \approx 2.7183$.]