Exam: Analysis-I (202001329) MOD-02-AM: Mathematical Proof Techniques

18-April-2023, 08:45 – 11:45 Total Points : 40

Instructor: Pranab Mandal

- Closed book exam! Use of own text-material or an electronic calculator is not allowed.
- All answers must be motivated. For the short-answer questions (1-2), a brief one is enough.
- For the other questions (3-5), you *must* follow the four steps (practiced during TBL).
  (i.) Get Started: describe what the problem is [and your initial thoughts]
  (ii.) Devise Plan: provide an outline of how you plan to solve (or have solved) the problem
  (iii.) Execute: execute your plan (and try) to reach your solution
  (iv.) Evaluate: reflect on your solution and/or approach [with something new not yet mentioned]

  Points are distributed (roughly) as: steps (i.)+(ii.) 40%, step (iii.) 40% and step (iv.) 20%. Exact allocations are mentioned next to each (part of a) question.
- Good Luck!

[Short-answer questions]

1. Consider the following set
$$A = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}.$$
Show that A does not have a minimum. [4]

2. Suppose that $a, b \in \mathbb{R}$, $a < b$, $n \in \mathbb{N}$, and $f : [a, b] \to \mathbb{R}$ is continuous. Consider the statement:
$$\text{If } \int_a^b x^n f(x)\, dx = 0, \text{ then } f(x) = 0 \text{ for at least one } x \in [a, b].$$
Prove that the statement above is true if n is even, but it is not true if n is odd. [4+2]

[Four-step questions]

3. Answer the following questions concerning limits of real-valued functions. [5+6+1]

   (a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is such that $\lim_{x\to a} f(x)$ exists for $a \in \mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers converging to a as $n \to \infty$, then $\lim_{n\to\infty} f(x_n)$ exists.

   (b) Show that the function f defined as
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
has no limit as $x \to 0$.

4. Suppose that the function f is defined on the interval $(0, 1)$ as

$$f(x) = x \ln\left(\frac{1}{x}\right), \quad \text{for } x \in (0, 1).$$

Prove that f is uniformly continuous. [3+4+1]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a two times differentiable function with $f''(x) < 0$, for all $x \in \mathbb{R}$. Prove that [4+4+2]

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2}, \quad \forall\, x_1, x_2 \in \mathbb{R}.$$

[As part of the evaluation step, you must mention at least two related (but different) insights/results not yet mentioned in your previous steps.]

Grade:	$\frac{\text{obtained score}}{40} \times 9 + 1$