UNIVERSITEIT TWENTE.

Date : April 3, 2012
Reference : DE.11-12(1)
Number of pages : 2

Differential Equations

Monday April 16th, 2012, 8:45 - 11:45 uur
Location: SC 0
Course code: 191560380

All answers must be clarified.

It is allowed to use an electronic calculator.

Cell phones must be turned off at all times!

1. Solve the following initial value problem:

$$\begin{cases} (2x+y)\,dx + (x+2y)\,dy = 0, \\ y(0) = 1. \end{cases}$$

2. The matrix A is defined by

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

   (c) Find e^{tA}.

   (d) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y},$$
$$\mathbf{y}(0) = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

3. Consider the following system of differential equations:

$$x' = y^2 - x^2,$$
$$y' = 2 - x^2 - y^2.$$

(a) Find all equilibrium points.

(b) Classify each equilibrium point: is it a saddle point, a nodal source or sink or a spiral source or sink?

(c) Find the nullclines and sketch them in the phase plane.

(d) Skecth arrows across the nullclines that indicate the direction of the trajectories crossing the nullclines.

4. Consider the system

$$
\begin{aligned}
x' &= 2y + x\,\sin(xy),\\
y' &= -2x - y\,\sin(xy).
\end{aligned}
$$

(a) Show that the system is Hamiltonian, without explicitly calculating the Hamiltoninan function.

(b) Find a Hamiltonian function of the system.

5. Consider the system

$$
\begin{aligned}
x' &= xy - xy^2,\\
y' &= -y^3 - 2x^2.
\end{aligned}
$$

Define $V(x, y) = \alpha x^2 + y^2$, where α is a positive constant.

(a) Find $\dot{V}$.

~~(b) Find a value of α for which $\dot{V}$ is negative semi-definite at $(0, 0)$.~~

(c) What can you say about the behavior of the equilibrium point $(0, 0)$: is it unstable, stable or asymptotically stable?

6. The wave equation for the displacement $u(x, t)$ of a string with velocity constant $c = 1$ is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

Use Fourier series to find $u(x, t)$ for a string with length $L = \pi$ with fixed endpoints, initial displacement $u(x, 0) = 0$, and initial velocity

$$u_t(x, 0) = \frac{\pi}{2} - x \quad \text{for } 0 \leq x \leq L.$$

Awarded points

2	8	8dg	3		
1 : 3	2 a: 2 b: 2 c: 3 d: 2	3 a: 1 b: 3 c: 2 d: 3	4 a: 2 b: 3	5 a: 2 b: 2 c: 1	6 : 5

Total: $36 + 4 = 40$ points.