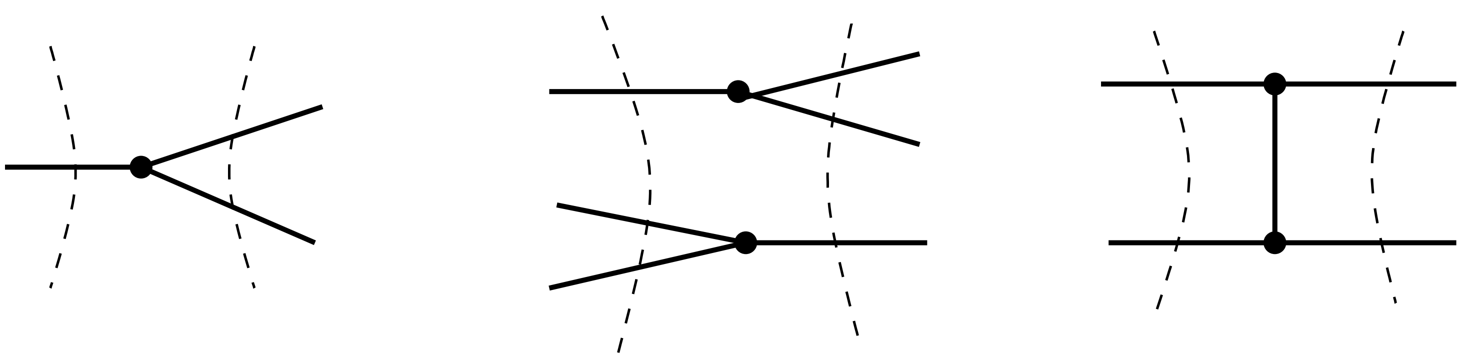
**Solutions Sample Exam**

1. **a)** is a degree sequence. Draw the corresponding graph. (Vertices $1-4$ induce a $K_4$.)
   **b)** Vertices $1, 2, 3, 4$ must have at least 7 edges (two from each degree 5 vertex, one from vertex 4) joining them to vertices $5, 6, 7, 8$. The latter have total degree only 5. Contradiction.

2. Cases $\kappa = 0$ and $\kappa = 3$ are trivial. (The latter because $\kappa \le \kappa' \le \delta = 3$).

   If $\kappa = 1$, then let $v$ be a cut vertex whose removal splits $G$ into two (or more) components. Then in $G$, one of these components is joined to $v$ by only one edge (as $d_v = 3$). Removing this edge disconnects the graph (cf. figure below).

   If $\kappa = 2$, let $\{u, v\}$ be a 2-vertex cut whose removal splits $G$ into two (or more) components. Assume first that $u$ and $v$ are non-adjacent. Then, as in the previous case, each of $u$ and $v$ is joined to one of the components by only one edge. Removing these two edges disconnects the graph. Finally, assume that $u$ and $v$ are adjacent. Then $u$ and $v$ are each joined to both components by only one edge. Removing the two edges joining $u$ and $v$ to the same component disconnects the graph.



3. Group the $2k$ odd degree nodes into $k$ pairs (in an arbitrary way). Connect the two nodes of each pair by an additional edge. (If you want to avoid multiple edges you may connect them also by paths of length $\ge 2$.) The resulting graph has a eulerian tour. Removing the inserted edges (resp. paths) from the tour gives the desired path partition $E = P_1 \cup ... \cup P_k$.

4. Tutte's Thm: $G = (V, E)$ has a perfect matching iff
   $o(G \backslash S) \le |S|$ for every $S \subseteq V$. $\qquad$ ($o(G)$ = number of odd components of $G$.)

   More generally, $\max o(G \backslash S) - |S|$ (with max taken over all sets $S \subseteq V$) provides a tight lower bound on the number of nodes that must remain unmatched by any matching. In other words,

$$\max |M| = \frac{1}{2} \min\{n - (\max o(G\backslash S) - |S|)\}$$

.

5. Vizing states that $\chi' \in \{k, k+1\}$. Assume that $\chi' = k$. Then $E$ is a disjoint union of $k$ perfect matchings. (Each matching consisting of the edges of some some color $i$, $i = 1, ..., k$.)
   Now consider a cut vertex $v$ and the perfect matching consisting of all edges of color 1. One of these edges joins $v$ to a component of $G\backslash v$. The remaining nodes of this component are completely matched (pairwise) with edges of color 1. We conclude that this must be an odd component. All other components of $G\backslash v$ must be even (as these have perfect matchings of color 1). However, the same argument, applied to an edge of color $i$ joining $v$ to another component of $G\backslash v$ shows that this other component must be odd, a contradiction.

6. We prove the following slightly more general claim for a graph on $n$ vertices:
   If $H$ is a subgraph of $G$ and $H'$ is a subgraph of $G^c$, then $\chi(H) + \chi(H') \leq n + 1$.

   Proof: Assume to the contrary that $\chi(H) + \chi(H') > n + 1$. W.l.o.g. $H$ and $H'$ are critical. Say, $H$ is $k$-critical and $H'$ is $k'$- critical with $k + k' > n + 1$. Then $H$ contains at least $k$ nodes and has min degree $\delta \geq k - 1$. Similarly, $H'$ has at least $k'$ nodes and min degree $\delta' = k' - 1$. As $k + k' > n$, there exists a node $v$ that $H$ and $H'$ have in common. So its degree in $G$ is at least $k - 1$ and its degree in $G^c$ is at least $k' - 1$. Thus $n - 1 = d_G(v) + d_{G^c}(v) \geq k - 1 + k' - 1 > n - 1$, a contradiction.

7. Draw an edge (straight line segment) between any two points at distance exactly 1. No two of these intersect each other in an interior point. (For such an intersection point would divide each edge into two parts, one of which has length at most $\frac{1}{2}$. As a consequence, two of the points were at distance $< \frac{1}{2} + \frac{1}{2} = 1$, contrary to the assumption.)
   Hence the edges of length 1 define a plane graph $G$ with $n$ nodes, $e$ edges and $f$ faces. Such a graph has $e \leq 3n - 6$. (If you add edges until the graph is a triangulation, you end up with $e = 3n - 6$.)

**Points: 36+4 = 40**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1: 5 | 2: 5 | 3: 5 | 4: 5 | 5: 5 | 6: 6 | 7: 5 |