Optimal Control
(course code: 191561620)

Date: 05-04-2015
Place: CR-3H
Time: 08:45-11:45

1. Consider the nonlinear system
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 - x_1 x_3 \\ -x_3 + x_1 x_2 \end{bmatrix}. \tag{1}$$
   (a) Determine all points of equilibrium.
   (b) Consider equilibrium $\bar{x} = (0,0,0)$. What does candidate Lyapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$ allow us to conclude about the stability properties of this equilibrium?

2. Consider
$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - 2x_2 \end{aligned}$$
with equilibrium $\bar{x} = (0,0)$. Determine a Lyapunov function $V(x)$ such that $\dot{V}(x) = -2x_1^2 - x_2^2$ and verify that this $V(x)$ is a strong Lyapunov function for this system.

3. Consider minimizing
$$\int_0^1 x(t)\dot{x}(t)^2 \,\mathrm{d}t,$$
over all functions $x(t)$ subject to $x(0) = 4, x(1) = 1$.
   (a) Argue that if the optimal solution $x(t)$ is nonnegative for all $t \in [0,1]$ and $x(0) = 4, x(1) = 1$ that then $\dot{x}(t) \le 0$ for all $t \in [0,1]$.
   (b) Which function $x(t) \ge 0, \dot{x}(t) \le 0$ solves the Beltrami identity and satisfies the boundary conditions $x(0) = 4, x(1) = 1$?

   [Hint: you may want to use that $x^\gamma(t)\dot{x}(t) = a$ iff $\frac{x^{\gamma+1}(t)}{\gamma+1} = at + b$ whenever $x(t) > 0$ and $\gamma \ne -1$.]
   (c) Is Legendre's second order condition satisfied?

4. Consider
$$\dot{x}(t) = -x(t) + u(t), \qquad x(0) = \mathrm{e}, \quad x(2) = 1.$$
This is a system with both initial and final constraint. We want to minimize
$$\int_0^2 |u(t)| \,\mathrm{d}t$$
with $u(t) \in [-1,1]$ for all $t \in [0,2]$. (Notice the absolute value in the cost function.)

(a) Determine the Hamiltonian H

(b) Express the optimal $u(t)$ in terms of the costate $p(t)$ and argue that at any moment in time we have either $u(t) = -1$ or $u(t) = 0$ or $u(t) = +1$

(c) Determine the costate equations and its general solution $p(t)$

(d) Show that if $u(0) \neq 0$ then $u(t)$ is constant over $t \in (0, 2]$.

(e) Determine the optimal input for the given initial and final constraint $x(0) = \mathrm{e}, x(2) = 1$

5. Consider the optimal control problem

$$\dot{x}(t) = u(t), \qquad x(0) = x_0$$

with $u(t) \in \mathbb{R}$ and cost

$$J_{[0,\infty)}(x_0, u(\cdot)) = \int_0^1 u^2(t)\,\mathrm{d}t + \int_1^\infty 4x^2(t) + u^2(t)\,\mathrm{d}t.$$

(a) Assume first that $x(1)$ is given. Determine the optimal cost-to-go from $t = 1$ on: $V(x(1), 1) := \min_u \int_1^\infty 4x^2(t) + u^2(t)\,\mathrm{d}t$.

(b) Express the optimal cost $J_{[0,\infty)}(x_0, u(\cdot))$ as $J_{[0,\infty)}(x_0, u(\cdot)) = \int_0^1 u^2(t)\,\mathrm{d}t + Sx^2(1)$. (That is: what is S?)

(c) Solve the optimal control problem: determine the optimal cost $J_{[0,\infty)}(x_0, u(\cdot))$ and express the optimal input $u(t)$ as a function of $x(t)$. [Hint: see the hint of problem 3.(b)].

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problem:	1	2	3	4	5
points:	2+3	4	2+4+2	1+3+2+2+2	3+2+4

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange:

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H(x, p, u),}{\partial p} \qquad x(0) = x_0,$$

$$\dot{p} = -\frac{\partial H(x, p, u),}{\partial x}, \qquad p(T) = \frac{\partial S(x(T))}{\partial x}$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = S$$

Hamilton-Jacobi-Bellman:

$$\frac{\partial V(x, t)}{\partial t} + \min_{u \in \mathbb{U}}\left[\frac{\partial V(x, t)}{\partial x^T} f(x, u) + L(x, u)\right] = 0, \qquad V(x, T) = S(x)$$

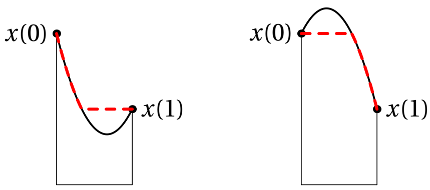
1. (a) only $(0,0,0)$

   (b) For "ease of exposition" denote (x_1, x_2, x_3) as (x, y, z). The V is continuously differentiable and positive definite. $\dot{V}(x) = 2x(-x+y) + 2y(x-y-xz) + 2z(-z+xy) = 2[-xx+xy+xy-yy-xyz+-zz+xyz] = -2(xx-2xy+yy+zz) = -2(x-y)^2 - 2z^2$ so it is ≤ 0 but not < 0 (for $z=0, x=y \neq 0$). So stable but perhaps not asymptotically stable.

   (You don't have to invoke LaSalle but if you do then you'll see that it *is* in fact asymptotically stable.)

2. the linear equation $PA + A'P = \left[\begin{smallmatrix} -2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ gives $P = \frac{1}{6}\left[\begin{smallmatrix} 5 & 1 \\ 1 & 2 \end{smallmatrix}\right]$. Since $p_{11} = 5/6 > 0$ and $\det(P) = 1/4 > 0$ this matrix is positive definite, so $V := x'Px > 0, \dot{V} < 0$ and thus V is a strong Lyapunov function.

3. (a) Suppose $x(t)$ has a positive derivative, then connecting the local maxima/minima such as here in red

   makes $\dot{x} = 0$ on these regions so makes $\int x\dot{x}^2$ smaller. The optimal x hence has no local maxima/minima.

   (b) Beltrami says $C = x\dot{x}^2 - (x2\dot{x})\dot{x} = -x\dot{x}^2$. So $x^{1/2}\dot{x} = a$ for some constant a. By the hint this means $x^{3/2}(t) = (3/2)(at+b)$. Initial condition then becomes $4^{3/2} = (3/2)b$ so $8 = (3/2)b$ so $b = 16/3$. Final condition: $1^{3/2} = 3/2(a+b)$ so $a = 2/3 - b = -14/3$. That is $x(t) = (8-7t)^{2/3}$.

   (c) Yes: $\frac{\partial^2 F(t,x(t),\dot{x}(t))}{\partial \dot{x}^2} = 2x(t) \geq 0$ for all $t \in [0,1]$.

4. (a) $H = p(-x+u) + |u|$

   (b) If $p > 1$ then $p(-x+u)+|u|$ is minimal for $u = -1$. If $p < -1$ then $p(-x+u)+|u|$ is minimal for $u = +1$. If $-1 < p < 1$ then $p(-x+u)+|u|$ is minimal for $u = 0$:

   $$u(t) = \begin{cases} -1 & \text{if } p(t) > 1 \\ 0 & \text{if } |p(t)| < 1 \\ +1 & \text{if } p(t) < -1 \end{cases}$$

   (c) $\dot{p} = p$ without final condition (because there is a final condition on x). General solution is $p(t) = c\,\mathrm{e}^t$

   (d) If $u(0) \neq 0$ then $u(t) = \pm 1$ so $\mp p(0) \geq 1$. Since $p(t) = p(0)\,\mathrm{e}^t$ and e^t increases we have that then $\mp p(t) > 1$ for all $t > 0$, so $u(t) = \pm 1$ for all $t > 0$

   (e) A bit tricky to explain:

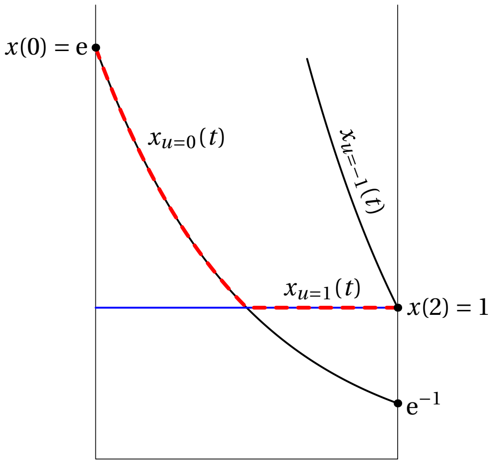
part (d) says that $u(0) = \pm 1$ implies $u(t) = \pm 1$ for all $t \in [0,1]$. These are not feasible:

- If $u(t) = 1$ for all time then $\dot{x} = -x + u, x(0) = \mathrm{e}$ gives $x(t) = 1 + (x(0) - 1)\,\mathrm{e}^{-t} > 1$ for all time so not $x(2) = 1$.
- If $u(t) = -1$ for all time then $\dot{x} = -x + u, x(0) = \mathrm{e}$ gives $x(t) = -1 + (x(0) + 1)\,\mathrm{e}^{-t}$ for all time so not $x(2) = 1$.

Hence $u(t)$ must be zero initially, so $|p(0)| < 1$. As time increases the value $|p(t)|$ might become 1 at some time t_0. For $t > t_0$ the value of $u(t)$ must then be $+1$ or -1 for the rest of time. Since we need to end up at $x(2) = 1$ this mean that if $t_0 < 2$:

$$x_{u=1}(t) = \underbrace{1 + (x(2) - 1)\,\mathrm{e}^{2-t}}_{1}, \qquad x_{u=-1}(t) = -1 + (x(2) + 1)\,\mathrm{e}^{2-t}$$

for all $t \in [t_0, 2]$. See the plot:



Clearly the only possible solution is the red one: for $t \in [0,1]$ we have $u(t) = 0$ and for $t \in [1,2]$ we have $u(t) = +1$.

5. (a) The Algebraic Riccati becomes $0 = P^2 - 4$. So $P = 2$: $V(x(1), 1) = 2x(1)^2$

   (b) The principle of optimality says that $Sx^2(1) = V(x(1), 1) = Px^2(1)$. So $S = P = 2$

   (c) The Riccati differential equation becomes

   $$\dot{P} = P^2, \qquad P(1) = 2.$$

   This is of the form $P^\gamma \dot{P} = a$ for $\gamma = -2$ and $a = 1$. so the hint of the hint says that $P^{-1}(t)/(-1) = t + b$ so $P(t) = 1/(-b - t)$. Given that $P(1) = 2$ it follows that $b = -3/2$, so

   $$P(t) = \frac{1}{3/2 - t}, \qquad t \in [0,1].$$

   The optimal cost hence is $x^2(0)P(0) = \frac{2}{3}x_0^2$ and $u(t) = -P(t)x(t)$ for $t \in [0,1]$ and $u(t) = -2x(t)$ for $t > 1$.