Optimal Control
(course code: 156162)

Date: 21-06-2011
Place: 13:45–16:45
Time: HB-2B

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -2x_2 + 2x_1x_2 \end{bmatrix}$$

   (a) Determine all points of equilibrium.
   (b) Is the linearization at equilibrium $\bar{x} = (2, 0)$ asymptotically stable?
   (c) Argue that $\{(x_1, x_2) \mid x_2 = 0\}$ is an invariant set.
   (d) Is the *nonlinear* system at equilibrium $\bar{x} = (2, 0)$ asymptotically stable?

2. Determine all $\alpha, \beta \in \mathbb{R}$ for which the matrix

$$P = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & \beta \\ 0 & \beta & 2 \end{bmatrix}$$

   is positive definite.

3. One theorem in Chapter 1 on Lyapunov stability allows to test for *global* asymptotic stability. Formulate this theorem.

4. Consider minimizing the cost function

$$J := \int_0^1 \tfrac{1}{2}x^2(t) + \tfrac{1}{2}(\dot{x}(t) + x(t))^2 \, dt$$

   with initial $x(0) = x_0$ and free end-point $x(1)$.

   (a) Determine the Euler equation for this problem.
   (b) Determine the optimal $x(t)$.

5. Consider the linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   with cost function

$$J := -\frac{1}{4}x_1(T) + \int_0^T \tfrac{1}{2}u^2(t) \, dt.$$

   We allow any input-value: $u(t) \in \mathbb{R}$ for all $t \in [0, T]$.

   (a) Determine the Hamiltonian equations (the formulae for state x and co-state p).
   (b) Find the optimal co-state and the optimal input.
   (c) Now suppose that the input is restricted to $u(t) \in [-1, 1]$ for all time. Find the optimal input.

6. Consider the system with bounded input

$$\dot{x}(t) = u(t), \qquad x(0) = x_0, \qquad u(t) \in [-1, 1]$$

on the finite time horizon $t \in [0, T]$ with cost function

$$J = x^2(T).$$

(a) Argue that the optimal input is

$$u(t) = \begin{cases} +1 & \text{if } x(t) < 0 \\ 0 & \text{if } x(t) = 0 \\ -1 & \text{if } x(t) > 0 \end{cases}$$

(b) Determine the value function $V(x, t)$.
[Hint: use the definition of value function and the optimal $u(t)$.]

(c) Verify that your $V(x, t)$ satisfies the Belmann equation.

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problem:	1	2	3	4	5	6
points:	3+3+2+2	3	3	2+4	3+3+2	2+4+2

Exam grade is $1 + 9p/p_{\text{max}}$.

Euler:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}(x, p, u), & x(0) &= x_0, \\ \dot{p} &= -\frac{\partial H}{\partial x}(x, p, u), & p(t_e) &= \frac{\partial S}{\partial x}(x(t_e)) \end{aligned}$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(t_e) = G$$

Bellman:

$$\frac{\partial W}{\partial t}(x, t) + \min_{v \in \mathcal{U}} \left[\frac{\partial W}{\partial x^T}(x, t) f(x, v) + L(x, v)\right] = 0, \qquad W(x, t_e) = S(x)$$