Theory of Partial Differential Equations **SOLUTIONS**

Final Exam (155010)

2014.01.29

01. Consider the first order, quasilinear partial differential equation

$$-\mathbf{y}\,\mathbf{U}_{\mathbf{x}} + \mathbf{x}\,\mathbf{U}_{\mathbf{y}} = \mathbf{x}\,\mathbf{U}\,. \tag{1}$$

(a) Determine the characteristics of (1).
(b) Find the solution to (1) corresponding to the data $\mathrm{U}(\mathbf{0},\mathbf{y}) = \mathbf{1}$, if it exists. (If it does not, explain why not.)

Solution. The PDE is first order, which points to the method of characteristics. Since it is also quasilinear, we will formulate the characteristic equations in $(x, y, u)-$space:

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x, \\ \dot{u} &= xu. \end{aligned}$$

The top subsystem for (x, y) is closed (i.e. does not involve u), so it can be solved independently:

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x, \end{aligned} \quad \text{yielding } \big(x(t), y(t)\big) = \big(C_1 \cos t + C_2 \sin t\ ,\ C_1 \sin t - C_2 \cos t\big).$$

[Here, I take the solution for granted; you are expected to derive it, instead. This is nothing other than a harmonic oscillator in phase space, of course.] Substituting into the ODE for u, we find

$$\dot{u} = xu = (C_1 \cos t + C_2 \sin t)\, u,$$

which is separable:

$$\int^u \frac{\mathrm{d}\bar{u}}{\bar{u}} = \int^t x(\bar{t})\,\mathrm{d}\bar{t} = \int^t \big(C_1 \cos\bar{t} + C_2 \sin\bar{t}\big)\,\mathrm{d}\bar{t}, \quad \text{whence } u(t) = C_3\,\mathrm{e}^{C_1 \sin t - C_2 \cos t}.$$

The characteristics are, thus, the collection of curves

$$\big(x(t), y(t), u(t)\big) = \Big(C_1 \cos t + C_2 \sin t\ ,\ C_1 \sin t - C_2 \cos t\ ,\ C_3\,\mathrm{e}^{C_1 \sin t - C_2 \cos t}\Big),$$

with (C_1, C_2, C_3) arbitrary constants.

To incorporate the boundary data, we parameterize it as $\{(0, s, 1)\,|\,s\}$ and demand that the characteristic curves above lie on it at time zero:

$$\begin{aligned} x(0) &= C_1 &&= 0, \\ y(0) &= -C_2 &&= s, \\ u(0) &= C_3\,\mathrm{e}^{-C_2} &&= 1, \end{aligned} \quad \text{with solution} (C_1, C_2, C_3) = \Big(0, -s, \mathrm{e}^{-s}\Big).$$

It follows that the solution surface may be expressed parametrically as

$$\begin{aligned} x(t,s) &= -s \sin t, \\ y(t,s) &= s \cos t, \\ u(t,s) &= \mathrm{e}^{-s}\mathrm{e}^{s \cos t}. \end{aligned}$$

Remark. The PDE is actually linear, so it can also be solved by the method of characteristics on the $(x, y)-$plane. This was evident above, already, in that the ODE system for (x, y) decouples from u. The characteristics on the $(x, y)-$planes are obviously circles: recall that we are dealing with a harmonic oscillator on the phase plane or else note that $x^2 + y^2 = const.$ along characteristics. In polar coordinates, our BVP becomes

$U_\theta = r \cos\theta\, U$, subject to $U(r, \pi/2) = 1$.

Plainly, the solution is

$U(r, \theta) = \mathrm{e}^{r(\sin\theta - 1)}$ or, with a slight abuse of notation, $U(x, y) = \mathrm{e}^{y - \sqrt{x^2+y^2}}$.

02. Solve for U the following initial–boundary value problem for the diffusion equation:

$$\begin{aligned} \mathbf{U_t(x,t)} &= \mathbf{U_{xx}(x,t)}\,, && \textbf{for all}\quad \mathbf{0 < x < 1} \quad \textbf{and} \quad \mathbf{t > 0}\,, \\ \mathbf{U(x,0)} &= \sin\left(\frac{\pi}{\mathbf{2}}\,\mathbf{x}\right) + \mathbf{3}\,, && \textbf{for all}\quad \mathbf{0 \le x \le 1}\,, \\ \mathbf{U(0,t)} &= \mathbf{1} \quad \textbf{and} \quad \mathbf{U_x(1,t) = 0}\,, && \textbf{for all}\quad \mathbf{t > 0}\,. \end{aligned} \tag{2}$$

[Note: Derive the eigenvalues $\lambda_\mathbf{n}$ and eigenfunctions X_n explicitly; do not just copy them from your notes/the book. You may, nevertheless, assume that $\lambda_\mathbf{n} \le 0$ for all $\mathbf{n}$: that is, no positive eigenvalues exist.]

Solution. The first and crucial observation is that the problem does not satisfy homogeneous BCs, since $U(0, t) = 1$. This implies that we must locate and subtract the steady state, which solves the problem

$U''(x) = 0$, subject to $U(0) = 1$ and $U'(1) = 0$;

the solution is $U^*(x) = 1$. The deviation $V(x, t) = U(x, t) - U^*(x)$ from the steady state satisfies, then, the problem

$$\begin{aligned} V_t(x,t) &= V_{xx}(x,t)\,, && \text{for all}\quad 0 < x < 1 \quad \text{and} \quad t > 0\,, \\ V(x,0) &= \sin\left(\frac{\pi}{2}\,x\right) + 2\,, && \text{for all}\quad 0 \le x \le 1\,, \\ V(0,t) &= 0 \quad \text{and} \quad V_x(1,t) = 0\,, && \text{for all}\quad t > 0\,. \end{aligned} \tag{3}$$

The eigenvalue problem is

$X'' = \lambda X$, subject to $X(0) = X'(1) = 0$.

Using e.g. complex exponentials to solve the ODE, we find the equation

$$\det \begin{vmatrix} 1 & 1 \\ i\sqrt{-\lambda}\mathrm{e}^{i\sqrt{-\lambda}} & -i\sqrt{-\lambda}\mathrm{e}^{-i\sqrt{-\lambda}} \end{vmatrix} = 0, \quad \text{whence } \lambda_n = -(n - 1/2)^2\pi^2,\ n = 0, \pm1, \pm2, \ldots.$$

The corresponding eigenfunctions are

$$X_n(x) = \sin\Big((n - 1/2)\pi x\Big);$$

since $X_0 = X_1$, $X_{-1} = X_2$, $X_{-2} = X_3$ et cetera, we can restrict n to be natural:

$$\lambda_n = -(n - 1/2)^2\pi^2 \quad \text{and} \quad X_n(x) = \sin\Big((n - 1/2)\pi x\Big), \quad \text{with } n = 1, 2, 3, \ldots.$$

The case $\lambda = 0$ must be treated apart. It yields no eigenfunction, hence $\lambda = 0$ is not an eigenvalue.

The general solution to the BVP satisfied by the deviation is, then,

$$V(x, t) = \sum_{n\ge1} C_n T_n(t) X_n(x), \quad \text{where } T_n(t) = \mathrm{e}^{\lambda_n t}.$$

Using the IC, we find

$$\sum_{n\ge1} C_n X_n(x) = V(x, 0) = \sin(\pi x/2) + 2\,.$$

The precise values of the coefficients are hard to guess, so one must resort to the Fourier formulas. Once the constants C_n have been found, the solution to the original problem is

$$U(x,t) = U^*(x) + V(x,t) = 1 + \sum_{n\geq 1} C_n \mathrm{e}^{-(n-1/2)^2\pi^2 t} \sin\Big((n-1/2)\pi x\Big).$$

03. Consider the following initial–boundary value problem for the wave equation on the half-line:

$$\begin{aligned} \mathbf{U_{tt}(x,t)} &= \mathbf{U_{xx}(x,t)}, && \textbf{for all} \quad \mathbf{x \geq 0} \quad \textbf{and} \quad \mathbf{t > 0}, \\ \mathbf{U(x,0)} &= \mathbf{f(x)}, && \textbf{for all} \quad \mathbf{x \geq 0}, \\ \mathbf{U_t(x,0)} &= \mathbf{0}, && \textbf{for all} \quad \mathbf{x \geq 0}, \\ \mathbf{U(0,t)} &= \mathbf{0}, && \textbf{for all} \quad \mathbf{t > 0}. \end{aligned} \tag{4}$$

(a) Solve the problem for $\mathbf{f(x) = \sin(x)}$.
(b) Let $\mathbf{x_* > 0}$ be arbitrary but fixed. Find all functions $\mathbf{f}$ for which the displacement at $\mathbf{x_*}$ is zero *in the long term*— that is, for which there exists a time instant $\mathbf{T}$ such that $\mathbf{U(x_*,t) = 0}$ for all $\mathbf{t \geq T}$.

Solution. The solution to the IBVP can be obtained by a simple application of the modified d'Alembert formula treated in the reader and elsewhere. One readily finds

$$U(x,t) = \begin{cases} \dfrac{1}{2}\big[f(x+ct) + f(x-ct)\big], & \text{for } t < x/c, \\ \dfrac{1}{2}\big[f(x+ct) - f(ct-x)\big], & \text{for } t \geq x/c. \end{cases}$$

Here in particular $c = 1$ and $f(x) = \sin x$, so

$$U(x,t) = \begin{cases} \dfrac{1}{2}\big[\sin(x+t) + \sin(x-t)\big], & \text{for } t < x, \\ \dfrac{1}{2}\big[\sin(x+t) - \sin(t-x)\big], & \text{for } t \geq x; \end{cases} \qquad \text{in other words, } U(x,t) = \sin x \cos t, \text{ for all } x,t \geq 0.$$

To answer the second part of the question, we must select a form for our solution, that is, whether to use the form for $t < x$ or for $t > x$. Since x_* is fixed and we are interested in arbitrarily large times, we focus on threshold times $T > x_*$ so as to use the second formula:

$$U(x_*,t) = \frac{1}{2}\big[f(x_* + t) - f(t - x_*)\big], \quad \text{for all } t \geq x_*.$$

The condition $U(x_*,t) = 0$ becomes, then,

$$f(t + x_*) = f(t - x_*), \quad \text{for all } t \geq T > x_*.$$

Since t is variable and x_* is a constant, this states nothing else than that f is $2x_*$−periodic. Less strongly, the (smallest) period of f may be $2x_*/n$, for some $n = 1, 2, 3, \ldots$. Additionally, it suffices that f has this property away from the origin, i.e. that $f(x + 2x_*) = f(x)$ for all $x > X$ (with some fixed X). In retrospect, this makes a lot of sense, both in light of part (a) and because of wave reflection at the boundary.

— SELECT AND SOLVE **ONLY ONE** OF THE FOLLOWING TWO PROBLEMS —

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04. Consider the following first-order problem on the plane:

$$\begin{aligned} -\mathbf{y\, U_x(x,y) + x\, U_y(x,y)} &= \mathbf{g(x,y)}, && \textbf{with} \quad \mathbf{(x,y) \in R^2}, \\ \mathbf{U(x,0)} &= \mathbf{f(x)}, && \textbf{with} \quad \mathbf{x \geq 0}. \end{aligned} \tag{5}$$

A smooth solution $U(x,y)$ to this problem does not always exist. Explain in detail why this is so, formulate conditions for f and g which guarantee that such a smooth solution $U(x,y)$ exists and derive an explicit

formula for it (possibly in another coordinate system).

[Note: You are not asked to formulate the best possible conditions under which U exists; exercise your judgement! Also, 'explain in detail' means that, ideally, you would submit a clearly—and cleanly—written, intelligent discussion of the issue at hand with a balance between the quantitative (formulas) and the qualitative (interpretation). In plain speak: neither a list of formulas without explanation nor wordy explanations without actual mathematics.]

Solution. This is a linear problem, so we can use the method of characteristics on the $xy-$plane. We already did that in our remark to problem 1 and found that polar coordinates suit the problem exceptionally well. In terms of these, the problem reads

$U_\theta(r,\theta) = G(\theta), \quad$ subject to $U(r,0) = F(r)$;

the connection between F, G and f, g are easy to derive. Plainly, the solution is

$$U(r,\theta) = U(r,0) + \mathrm{e}^{\int_0^\theta G(\phi)\mathrm{d}\phi} = F(r) + \mathrm{e}^{\int_0^\theta G(\phi)\mathrm{d}\phi}.$$

Any solution worth its salt in the classical sense must be differentiable. Nonetheless, depending on G, the solution above may not even be continuous. Indeed, we should obviously demand that $U(r,2\pi) = U(r,0) = F(r)$, since $\theta = 0$ and $\theta = 2\pi$ correspond to the same point on the $xy-$plane. This is only true if $\int_0^{2\pi} G(\phi) = 0$; this is easy to translate in terms of g.

Intuitively, our PDE prescribed the rate of change of the U along each circle centered at the origin; this is evident from its polar form. To make sure that the characteristic curves close, we must demand that the forcing term has zero mean along each such circle; this is precisely the condition above.

05. Let Ω be the region outside the unit disk centered at the origin:

$$\Omega = \left\{(x,y)\,|\,x^2+y^2>1\right\}.$$

Naturally, the boundary $\partial\Omega$ is the unit circle:

$$\partial\Omega = \left\{(x,y)\,|\,x^2+y^2=1\right\}.$$

Additionally, let U be the unique smooth and bounded solution to the following problem:

$$\begin{array}{rcll} U_{xx}(x,y) + U_{yy}(x,y) &=& 0, & \text{for all} \quad (x,y)\in\Omega, \\ U(x,y) &=& f(x,y), & \text{for all} \quad (x,y)\in\partial\Omega. \end{array} \tag{6}$$

Derive a formula for $U(x,y)$ by using any method you wish.

[*Hint: One way to proceed is by using our work in class and/or the book to rewrite (6) in polar coordinates (r,θ) and then working in the coordinate system $(s,\theta) = (1/r,\theta)$. If you need the expressions for $U_{xx}+U_{yy}$ in polar coordinates and/or Poisson's formula for harmonic functions on a disk, you may assume them to be known.*]

Solution. [This is a sketch; fill in the details.] The most expedient way to solve the problem is, indeed, to start from the Laplacian in polar form. Changing from (r,θ) to (s,θ) is a matter of differentiation (think chain rule). The Laplacian, expressed in (s,θ), turns out to have the same functional form with the Laplacian in polar form. Also, the domain has now been transformed to the unit disk. Hence, Poisson's formula applies and the solution is now known. Expressing it in terms of the original quantities is an easy exercise.