# Examination Partial Differential Equations

**course 155010, 15 November 2000**

1. Consider the linear first order PDE for $u$:

$$(1+t)u_t + \tfrac{1}{2}(1+x)u_x = -u \tag{1}$$

in the quarter plane $x > 0,\ t > 0$, subject to the condition

$$u(x,0) = \frac{1}{(1+x)^2}, \quad x > 0. \tag{2}$$

   (a) Determine the equations for the characteristics, and sketch a few solution curves.

   (b) In which part of the quarter plane, say $D$, is the solution determined by (2).

   (c) Determine $u(x,t)$ in $D$.

2. Consider the 1-D wave equation

$$u_{tt} = c^2 u_{xx} \tag{3}$$

for $t > 0$ and $x \in \mathbb{R}$, subject to the initial condition

$$u(x,0) = e^{-x^2}, x \in \mathbb{R}. \tag{4}$$

   (a) If in addition $u_t(x,0) = 0$, for all $x \in \mathbb{R}$, then show that the solution consists of two humps like (4), half as high, one running to the left and one to the right as $t$ increases.

   (b) Determine $\psi(x)$ such that the additional initial condition

$$u_t(x,0) = \psi(x) \tag{5}$$

   yields a solution that consists of only one hump running to the left, i.e. the solution is a function of $x + ct$ alone.

3. Given is the initial boundary value problem

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1,\ t > 0 \\ u(0,t) &= u(1,t) = 0, \quad t > 0 \\ u(x,0) &= \sin 3\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{6}$$

   (a) Solve this problem by separation of variables

   (b) Show that for all positive $t$

$$\int_0^1 u(x,t)^2 dx \le \frac{1}{2}. \tag{7}$$

(c) Formulate the weak maximum principle and give the consequence for problem (6).

4. The bounded solution of the elliptic problem on the *half* plane

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad -\infty < x < \infty,\ y > 0 \\ u(x,0) &= f(x), \quad -\infty < x < \infty \end{aligned} \tag{8}$$

can be written in integral form as

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(\xi)}{y^2 + (\xi - x)^2}\, d\xi \tag{9}$$

Now consider the problem on the *quarter* plane

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad x > 0,\ y > 0 \\ u(x,0) &= h(x), \quad x > 0 \\ \frac{\partial u}{\partial x}(0,y) &= 0, \quad y > 0 \end{aligned} \tag{10}$$

Extend problem (10) to the half plane and make use of (9) to find an integral expression for the solution of (10).

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| 1. | a | 3 | 2. | a | 4 | 3. | a | 5 | 4. | 7 |
| | b | 2 | | b | 5 | | b | 2 | | |
| | c | 5 | | | | | c | 3 | | |