UNIVERSITEIT
TWENTE.

Applied Analysis Group
Dep. of Applied Mathematics
G.A.M. Jeurnink, C. Brune

Course: Applied Functional Analysis (191506302)

Date: Tuesday, Jan 26, 2016
Time: 8:45-11:45

- An explanation to every answer is required
- You can make use of a calculator
- (Part of) the scores to exercises **2**(c) and **5** may be earned by homeworks, see the underlined numbers in the table at the end of the exam.

---

1. Let the operator $T : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ be defined by:

$$\begin{aligned} Te_{2k-1} &= \frac{1}{k}e_{2k-1} + \frac{i}{k}e_{2k} \\ Te_{2k} &= -\frac{i}{k}e_{2k-1} + \frac{1}{k}e_{2k}, \quad k = 1, 2, \cdots \end{aligned}$$

where e_n denotes the n-th standard basis vector in $\ell^2(\mathbb{C})$.

(a) Compute the adjoint of T.

(b) Show that T is compact.

(c) Find all the eigenvalues of T.

(d) Compute the spectrum of T.

2. By $L^2(0, 2\pi)$ we denote the real vector space of all (classes of) real square-integrable functions on $(0, 2\pi)$, endowed with the usual inner product

$$(f, g) = \int_0^{2\pi} f(x)g(x)dx \ .$$

Let D be the closed linear subspace given by

$$D := \left\{ f \in L^2(0, 2\pi) \mid \int_0^{2\pi} f(x)dx = 0 \right\} \ .$$

(a) Determine $D^\perp$, the orthoplement of D in $L^2(0, 2\pi)$.

(b) Find the best approximation in D to $g(x) := x^2 + x$.

(c) Give a maximal orthonormal system (MOS) for the subspace D.

3. In the space of bounded real sequences, denoted by ℓ^∞, we define a linear subspace ℓ_0^∞ which consists of the sequences converging to zero.

(a) Show that the subspace ℓ_0^∞ is closed.

(b) The quotient space is given by $Q := \ell^\infty / \ell_0^\infty$ and equipped with the quotient space norm

$$||x + \ell_0^\infty||_Q := \inf \{ \ ||x + y||_{\ell^\infty} \mid y \in \ell_0^\infty \} \, .$$

Show that the dual space of this quotient space, Q^*, is nonzero.

4. Let $A : L^2[0,1] \to L^2[0,1]$ be the kernel operator $Af(x) := \int_0^1 k(x,y)f(y)dy$ with kernel function

$$k(x,y) := \begin{cases} y(1-x) & \text{for } 0 \le y \le x \le 1 \\ x(1-y) & \text{for } 0 \le x \le y \le 1\,. \end{cases}$$

It is known that the eigenvalues of A are the numbers $\lambda_n = \frac{1}{n^2\pi^2}$ with corresponding eigenfunctions

$$g_n(x) = \frac{1}{\sqrt{2}}\sin(n\pi x) \quad \text{for } n = 1, 2, 3, \cdots$$

(a) Prove that for a given $g \in L^2[0,1]$ the solution of

$$\begin{cases} u''(x) = -g(x) & \text{for } x \in (0,1) \\ u(0) = 0,\ u(1) = 0 \end{cases}$$

is given by $u = Ag$.

(b) Show that for $\lambda \in \mathbb{C}, \lambda \neq 0$ and given $g \in L^2[0,1]$ we have

$$\begin{cases} u'' + \lambda u = g \\ u(0) = 0,\ u(1) = 0 \end{cases} \iff (A - \frac{1}{\lambda}\mathrm{Id})u = \frac{1}{\lambda}Ag\ .$$

(c) Determine for which $\lambda \in \mathbb{C}$ the problem

$$\begin{cases} u'' + \lambda u = g \\ u(0) = 0,\ u(1) = 0 \end{cases}$$

has a unique solution for a given g, and describe this solution in terms of g and g_n.

5. Let F be a closed subspace of the Hilbert space H and $P \in BL(H)$ the orthogonal projection onto F.

(a) Given a Banach space G, show that the linear operator $T : H \to G$ is bounded, if and only if, the restrictions $T_{|F} : F \to G$ and $T_{|F^\perp} : F^\perp \to G$ are both bounded.

(b) Given a Banach space E, show that $T : E \to H$ is linear and bounded, if and only if, the compositions $P \circ T : E \to F$ and $(\mathrm{Id} - P) \circ T : E \to F^\perp$ are both linear and bounded.

Grading scheme:

1.	**2.**	**3.**	**4.**	**5.**
(a) 2	(a) 2	(a) 3	(a) 3	(a) 3
(b) 2	(b) 2	(b) 3	(b) 2	(b) 3
(c) 3	(c) 3		(c) 3	
(d) 2				

Total: $36 + 4 = 40$ points