Kenmerk: EWI07/TW/SK/016/EvD
Datum: 25 januari 2007

Exam Measure and Probability (157040)
Monday, 29 January 2007, 13.30 - 16.30 p.m.

This exam consists of 8 problems

1. a. Define what is meant by $m^*(A)$, the Lebesgue outer measure of a subset A of $\mathbb{R}$.

   b. Define what is meant by saying that $A \subset \mathbb{R}$ is measurable.

   c. Show that if $A \subset \mathbb{R}$ and $m^*(A) = 0$, then A is measurable.

2. Let Ω be a set, $\mathcal{F}$ a σ-field of subsets of Ω, and μ a $[0, \infty]$-valued function on $\mathcal{F}$. When do we call

   a. μ a measure?

   b. $(\Omega, \mathcal{F}, \mu)$ a probability space?

3. Suppose $E \subset \mathbb{R}$ is (Lebesgue-)measurable, and f and g are functions from E to $\mathbb{R}$.

   a. Define what is meant by saying that f is measurable.

   b. Show that the function $h(x) = \min\{f(x), g(x)\}$ is measurable if f and g are measurable.

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let X be a non-negative random variable with $0 < \mathbb{E}X = \int_\Omega X dP < \infty$. For $A \in \mathcal{F}$ define

$$P_X(A) = \frac{\int_A X dP}{\int_\Omega X dP}.$$

   Show that P_X is a probability measure on $(\Omega, \mathcal{F})$. (Hint: you might need the monotone convergence theorem at some point.)

5. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
   a. State the *dominated convergence theorem.*
   b. Evaluate
   $$\lim_{n\to\infty} \int_0^1 \frac{n\sin(x)}{(1+n^2x^{1/2})} dx.$$

6. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of
   a. the random variable X given by $X(\omega) = 1$ if ω is rational and $X(\omega) = 0$ otherwise;
   b. the random variable X given by $X(\omega) = \omega^2$.

7. Let m be Lebesgue measure and P a probability measure on $(\mathbb{R}, \mathcal{B})$ and define $F(x) := P((-\infty, x])$ and $G_c(x) := F(x+c) - F(x),\ x \in \mathbb{R}$.
   a. Show that, for any fixed $c > 0$, $\int_{\mathbb{R}} G_c dm = c$.
   b. Show that if F is continuous, then $\int_{\mathbb{R}} F dP = 1/2$.

   (Hint: use Fubini's theorem.)

8. Consider a sequence of functions $f_n(x) = n^2 e^{-n|x|},\ x \in \mathbb{R}$, and let $f(x) = 0,\ x \in \mathbb{R}$. Does f_n converge to f
   a. uniformly on $\mathbb{R}$?
   b. pointwise?
   c. almost everywhere?
   d. in L^p-norm?

1	2	3	4	5	6	7	8
3	2	2	2	2	2	3	2

Mark: $\frac{\text{Total}}{18} \times 9 + 1$ (rounded)