UNIVERSITEIT TWENTE

Faculteit Elektrotechniek, Wiskunde en Informatica

Exam Measurability and Probability (1915703401) on Monday, January 19, 2015, 8.45 – 11.45 uur.

The solutions of the exercises should be clearly formulated and clearly written down. Moreover, you should in all cases include a convincing argument with your answer.
With this exam a calculator is **not** permitted. Also a formula sheet is **not** permitted.

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1. Let $\Omega = \mathbb{R}$ and let $A_1, \dots, A_8$ be 8 disjoint, nonempty subsets of Ω. Let $\mathcal{F}$ be the smallest σ-algebra containing the sets $A_1, \dots, A_8$. Finally $\mu : \mathcal{F} \to \mathbb{R}^+$ is a measure.

   a) How many elements does $\mathcal{F}$ have? Clarify your computation!

   b) Let B be a subset of Ω. Show under what conditions on the set B, the indicator function 1_B is measurable with respect to the measure space $(\Omega, \mathcal{F}, \mu)$.

   Assume $A_i = [i, i+1)$ and $\mu(A_i) = i$. Finally, consider the function $f(x) = \lfloor x \rfloor$, i.e. $f(x)$ is the largest integer smaller than or equal to x.

   c) Check whether the integral:

   $$\int_0^5 f \, \mathrm{d}\mu$$

   is well-defined with respect to this measure space and, if so, compute the integral.

2. Consider the following expression:

   $$\lim_{n\to\infty} \int_0^{2\pi} \frac{\sin(nx)}{(x+1)(x+2)} \, \mathrm{d}x$$

   Is it possible to use the dominated convergence theorem to conclude convergence of this limit?

3. Investigate the convergence of:

   $$\lim_{n\to\infty} \int_0^1 \frac{\sqrt[n]{x}}{x + 1 - \sqrt[n]{x}} \, \mathrm{d}x$$

   and, if well-defined, compute the limit. If you use any theorems from the book then clearly formulate that theorem.

4. Consider the measure space $([0,3], \mathcal{M}_{[0,3]}, m_{[0,3]})$. Define the function:

   $$f(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{if } x \in [1,2) \\ 1 & \text{if } x \in [2,3) \end{cases}$$

   and consider two measures:

   $$\mu(A) = \int_A f \, \mathrm{d}m, \qquad \nu(A) = \int_A g \, \mathrm{d}m$$

a) Characterize under what conditions on the function g we have that $\nu \ll \mu$

b) Determine the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

5. Consider the probability space $(\Omega, \mathcal{F}, \mu)$ with $\Omega = [0,2]$, $\mathcal{F} = \mathcal{M}_{[0,2]}$ and $\mu = \frac{1}{2}m_{[0,2]}$. Let X and Y be random variables on the product space $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ defined by:

$$X(\omega_1, \omega_2) = \omega_1 \omega_2, \qquad Y(\omega_1, \omega_2) = \omega_1$$

a) Find the (cumulative) distribution function F_X

b) Compute $\mathbb{E}(X)$

c) Compute $P(X > Y)$

d) Compute $\mathbb{E}(X \mid Y)$

6. Consider the probability space $(\Omega, \mathcal{F}, \mu)$ with $\Omega = [0, \infty)$, $\mathcal{F} = \mathcal{M}_{[0,\infty)}$. Let μ be such that

$$\mu([a,b]) = \int_a^b \frac{1}{(x+1)^2} dm$$

Define two sequences of random variables:

$$X_n(\omega) = \begin{cases} \frac{1}{n} & n \le \omega \le 2n \\ 0 & \text{otherwise} \end{cases} \qquad Y_n(\omega) = \begin{cases} n & 0 \le \omega \le \frac{1}{n^2} \\ 0 & \text{otherwise} \end{cases}$$

Which of the following statements are true? (Justify your answers).

a) $Y_n \to 0$ in probability.

b) $Y_n \to 0$ almost surely.

c) $X_n \to 0$ pointwise.

d) $X_n \to 0$ in L_1-norm.

e) $Y_n \to 0$ in L_1-norm.

f) $X_n \to 0$ in L_2-norm.

g) $Y_n \to 0$ in L_2-norm.

For the questions the following number of points can be awarded:

Exercise	1.	11 points	8	Exercise	4.	8 points	8
Exercise	2.	5 points	5	Exercise	5.	12 points	4
Exercise	3.	7 points	5	Exercise	6.	11 points	11

The final grade is determined by adding 6 points to the total number of points awarded and dividing by 6.