UNIVERSITEIT TWENTE

Faculteit Elektrotechniek, Wiskunde en Informatica

Exam Measurability and Probability (1915703401) on Wednesday, April 12, 2017, 13:45 – 16:45 hours.

The solutions of the exercises should be clearly formulated and clearly written down. Moreover, you should in all cases include a convincing argument with your answer.
With this exam a calculator is **not** permitted. Also a formula sheet is **not** permitted.

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1. Consider the measurable space $(\Omega, \mathcal{B}, m)$ with $\Omega = [-\pi, \pi]$, $\mathcal{B}$ is the collection of Borel sets and m is the Lebesgue measure. Consider the three functions: $f_1, f_2, f_3 : \Omega \to \mathbb{R}$ defined by:

$$f_1(x) = x, \quad f_2(x) = \sin x, \quad f_3(x) = \operatorname{sgn} x = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

Let $\mathcal{F}_1$ be the smallest σ-algebra such that f_1 as a mapping from $(\Omega, \mathcal{F}_1)$ to $(\Omega, \mathcal{B})$ is measurable. We define $\mathcal{F}_2$ and $\mathcal{F}_3$ similarly as the smallest σ-algebras such that f_2 and f_3 respectively are measurable.

   a) Verify whether $\mathcal{F}_1 \subset \mathcal{F}_2$, $\mathcal{F}_1 = \mathcal{F}_2$ or $\mathcal{F}_1 \supset \mathcal{F}_2$.
   b) Verify whether $\mathcal{F}_1 \subset \mathcal{F}_3$, $\mathcal{F}_1 = \mathcal{F}_3$ or $\mathcal{F}_1 \supset \mathcal{F}_3$.
   c) Verify whether $\mathcal{F}_1 \subset \mathcal{B}$, $\mathcal{F}_1 = \mathcal{B}$ or $\mathcal{F}_1 \supset \mathcal{B}$.

2. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Investigate the convergence of:

$$\lim_{n\to\infty} \int_0^{12} \frac{n\cos(\frac{x}{n}) - x}{x + n}\, dm$$

If you use any theorems from the book then clearly formulate that theorem.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Investigate the convergence of:

$$\lim_{n\to\infty} \int_0^{\infty} n e^{-nx^2}\, dm$$

If you use any theorems from the book then clearly formulate that theorem.

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $\{E_n\}$ is a partition of Ω, i.e. $\Omega = \cup_{n=1}^{\infty} E_n$ where $\{E_n\}$ is a collection of pairwise disjoint measurable sets. Let $\{F_n\}$ be another partition of Ω which is a refinement of the partition $\{E_n\}$. Assume $\mu(E_n) < \infty$ and $\mu(F_n) < \infty$ for all $n \geq 1$. Consider two potential measures ν and ρ defined by:

$$\nu(A) = \sum_{n=1}^{\infty} \alpha_n \mu(A \cap E_n), \qquad \rho(A) = \sum_{n=1}^{\infty} \beta_n \mu(A \cap F_n).$$

   a) Find conditions on the α_n to characterize when ν is a measure on $(\Omega, \mathcal{F})$.
   b) Assume ν and ρ are both measures. Find conditions on the α_n, β_n to characterize when $\nu \ll \rho$.
   c) Assume ν and ρ are both measures with $\nu \ll \rho$. Determine the Radon-Nikodym derivative of ν with respect to ρ.

see reverse side

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5. Consider the probability space $(\Omega, \mathcal{F}, \mu)$ with $\Omega = [-1, 1]$, $\mathcal{F} = \mathcal{M}_{[-1,1]}$ and $\mu = \frac{1}{2}m_{[-1,1]}$. Let X and Y be random variables on the product space $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ defined by:

$$X(\omega_1, \omega_2) = \omega_1 \omega_2^2, \qquad Y(\omega_1, \omega_2) = 2\omega_1^2$$

   a) Find the (cumulative) distribution function F_X
   b) Compute $\mathbb{E}(X)$
   c) Compute $P(X > Y)$
   d) Compute $\mathbb{E}(X \mid Y)$

6. Consider the probability space $(\Omega, \mathcal{M}, m)$ with $\Omega = [0, 1]$, $\mathcal{M}$ the Lebesgue measurable subsets of Ω and m the Lebesgue measure. Define a sequence of random variables:

$$X_n(\omega) = \sqrt{n}\, \min\{\, 0, 2|n\omega - 1| - 1\,\}$$

   Which of the following statements are true? (Justify your answers).

   a) $X_n \to 0$ in probability.
   b) $X_n \to 0$ almost surely.
   c) $X_n \to 0$ pointwise.
   d) $X_n \to 0$ in L_1-norm.
   e) $X_n \to 0$ in L_2-norm.

For the questions the following number of points can be awarded:

Exercise	1.	9 points	Exercise	4.	9 points
Exercise	2.	9 points	Exercise	5.	9 points
Exercise	3.	9 points	Exercise	6.	9 points

The final grade is determined by adding 6 points to the total number of points awarded and dividing by 6.