Kenmerk: EWI2011/dmmp/001/BM

Exam Optimization Modeling (191581420)

Tuesday April 12, 2011, 8:45 – 11:45 h

- Use of calculators, mobile phones, etc. is not allowed!
- This exam consists of five problems. Please start a new page for every problem.
- Total number of points: 45 + 5 = 50. Distribution of points:

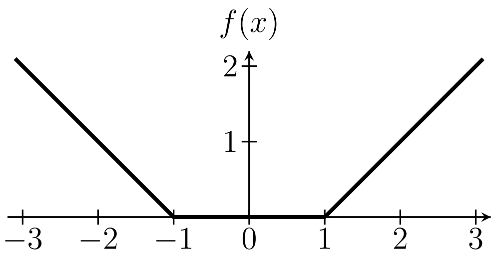
1a: 4	2a: 7	3a: 3	4a: 9	5a: 2
1b: 5	2b: 3	3b: 4	4b: 2	5b: 2
				5c: 2
				5d: 2

1. Modeling Tricks

Let

$$f(y) = \begin{cases} |y| - 1 & \text{if } |y| \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

This means that f looks as shown in the following graph.



(a) Write an LP for the following optimization problem:

$$\begin{aligned} &\text{minimize } f(x_1) \\ &\text{subject to } Ax \leq b. \end{aligned}$$

Show that your solution is correct. (An example does not suffice.)

(b) Write a MIP for the following optimization problem:

$$\begin{aligned} &\text{maximize } f(x_1) \\ &\text{subject to } Ax \leq b. \end{aligned}$$

Show that your solution is correct. (An example does not suffice.)

Note: The vector x is not restricted to be integer. You can assume that x_1 assumes only values within an interval $[-M, M]$ for some large M.

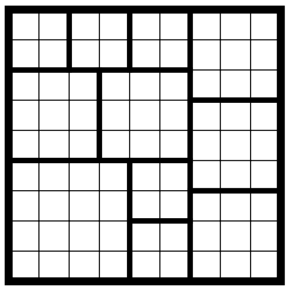
2. Square Packing

In the square packing problem, we are given a large square of side (integer) length W and a couple of small squares, each of which has a certain (integer) side length. The problem is: Cover as much area of the large square with the smaller squares.

Notes:

- You have just one copy of each small square available. But, of course, you do not have to use all squares.
- The small squares are not allowed to overlap.
- If you use a small square, then it must be completely within the large square.
- The large square consists of $W \times W$ *fields*. Any small square fills any field either completely or not at all.

Example: The following is a covering of a 9×9 square with squares of side lengths 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4. It is optimal because it covers the whole large square. Note that one square of side length 2 and one square of side length 4 is not used.

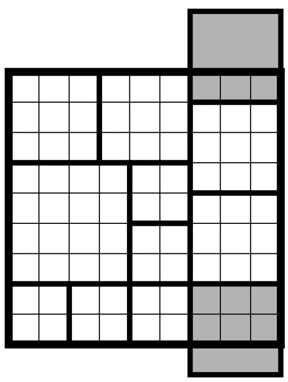


(a) Model the square packing problem as an IP.

Note: Efficiency is not important.

(b) Now assume that the square is in fact a torus. This means that the right-hand side is identical to the left-hand side and top and bottom are identical. This gives some more flexibility to position the small stones as they are allowed to "wrap around". Thus, also the solution shown below is feasible. (The two gray squares are in fact the same square that is wrapped around.)

Modify your model from Part (a) such that it handles this variant.



3. Knapsack

The *knapsack problem* is the following optimization problem: We are given a bunch of items $1, \dots, n$. Item i has value p_i and weight w_i. We can carry at most weight B in our knapsack. Thus, our goal is to maximize the sum of the values of the items:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} p_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i x_i \leq B, \\ & x_1, \dots, x_n \in \{0, 1\}. \end{aligned}$$

(a) Solve the following instance above using branch-and-bound: We have $W = 10$ and the three items are specified in the following table.

item	value	weight
1	3	4
2	5	5
3	9	6

(b) Now consider the following variant of the problem: There is an infinite number of copies per item. This means that we have to replace $x_1, \dots, x_n \in \{0, 1\}$ by $x_1, \dots, x_n \in \mathbb{N}$.

Solve this variant of the problem for the following instance using cutting planes. We have $W = 10$ and two items, which are specified in the following table.

item	value	weight
1	3	4
2	9	6

Note: First, solve the relaxation without tableaus. Second, construct the tableau, which is quite small because there is only one constraint. Do not forget the line for the objective function. Third, derive a cutting plane from the tableau. Fourth, try to remove the slack variable from the cutting plane. (This fourth step is not what your solver would do. But it simplifies things here very much.)

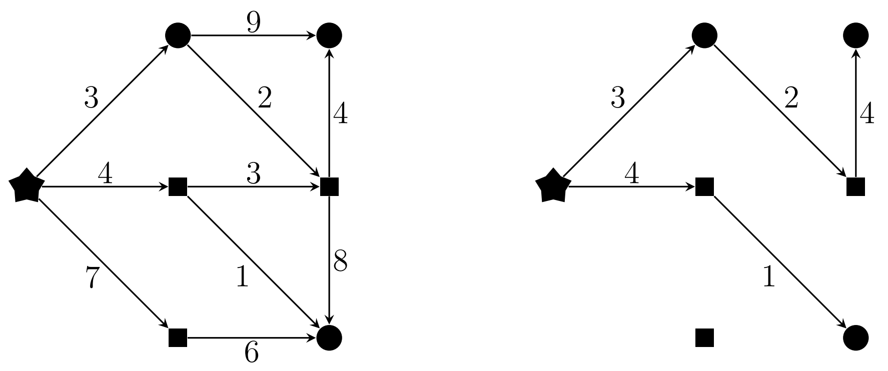
4. Distribution Network

Consider a computer network consisting of a number of computers. The computers are connected via links. These links are directed and have a bounded capacity.

Now consider the following problem: A video should be broadcasted from one root computer to a bunch of customer computers. Not all computers are customer computers. Every customer computer can receive the video but also route it through to other computers.

For every link you use you have to pay a certain non-negative amount that depends on the link but not on the amount of data you ship through it.

Example: In the network below, the root is drawn as a star, the customers are circles, all other computers are squares. The right-hand side shows an optimal solution, which has costs of 14.



(a) Build an IP model for this problem. Use the following sets (you do not have to use all indices, and if you need more, you can add some):

- u, v, w: computers
- e, f: links

(b) Now assume that the costs for link are proportional to the number of customer computers that receive the video through this link. For the solution on the right-hand side above, this means that the costs are now 17.

Describe briefly how you can model this variant of the problem. (No formal model is required.)

5. Questions

Answer the following questions and give a *short* justification for your answer.

(a) Consider the LP

$$\begin{aligned} &\text{minimize } y \\ &\text{subject to } x \leq 2, \\ &\qquad\qquad x, y \geq 0. \end{aligned}$$

Assume that you run the simplex method to solve this problem. Is it possible that simplex outputs the solution $x = 1$, $y = 0$?

(b) Consider the integer program

$$\begin{aligned} &\text{minimize } c^{\mathrm{T}}x \\ &\text{subject to } Ax = b, \\ &\qquad\qquad x \in \mathbb{N}^d. \end{aligned}$$

Let z_{IP} be the objective value of an optimum solution, and let z_{LP} be the objective value of the relaxation of this IP, i.e., when we replace $x \in \mathbb{N}^d$ by $x \geq 0$. (You can assume that both z_{IP} and z_{LP} exist, i.e., both LP and IP are neither unbounded nor infeasible.)

Which of the following cases can possibly occur?

(i) $z_{\mathrm{IP}} < z_{\mathrm{LP}}$?

(ii) $z_{\mathrm{IP}} = z_{\mathrm{LP}}$?

(iii) $z_{\mathrm{IP}} > z_{\mathrm{LP}}$?

(c) Consider again the IP of Question (b). Assume that $b \in \mathbb{Z}^n$ and that A is a totally unimodular.

Which of the following cases can possibly occur?

(i) $z_{\mathrm{IP}} < z_{\mathrm{LP}}$?

(ii) $z_{\mathrm{IP}} = z_{\mathrm{LP}}$?

(iii) $z_{\mathrm{IP}} > z_{\mathrm{LP}}$?

(d) Consider the LP

$$\begin{aligned} &\text{minimize } c^{\mathrm{T}}x \\ &\text{subject to } Ax = b, \\ &\qquad x \geq 0 \end{aligned}$$

and its dual

$$\begin{aligned} &\text{maximize } y^{\mathrm{T}}b \\ &\text{subject to } y^{\mathrm{T}}A \leq c^{\mathrm{T}}. \end{aligned}$$

Prove that for every feasible solutions x for the primal and y for the dual, the objective value of y is smaller than or equal to the objective value of x.