Kenmerk: EWI2016/dmmp/001/BM

Exam Optimization Modeling (191581420)

Friday, April 15, 2016, 8:45 – 11:45

- Use of calculators, mobile phones, etc. is not allowed!
- This exam consists of four problems. Start a new page for every problem.
- Total number of points: 45 + 5 = 50. Distribution of points according to the following table.

1a: 2	2a: 4	3a: 8	4a: 7
1b: 2	2b: 4	3b: 8	4b: 7
1c: 3			

1. Miscellaneous Questions

(a) *(2 points)* Model the constraint "$x \neq y$" for two integer variables x, y with range $\{0, \ldots, M\}$ using linear constraints (you are allowed to use additional variables).

(b) *(2 points)* Let a_0, a_1, a_2 be fixed numbers, and let x be an integer variable with range $\{0, 1, 2\}$. Model the constraint "$y = a_x$" using linear constraints (you are allowed to use additional variables).

(c) *(3 points)* Prove or disprove the following statement: For *every* matrix $A \in \mathbb{Z}^{n \times d}$ with $\mathrm{rank}(A) = n$, there *exists* a vector $b \in \mathbb{Z}^n$ with $b \neq 0$ such that the LP

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}} x \\ \text{subject to } & Ax = b, \\ & x \geq 0 \end{aligned}$$

has only integral basic feasible solutions.

Hint: Cramer's rule.

2. Modeling Bank Accounts

(a) *(4 points)* You have n bank accounts, numbered $1, \dots, n$. The current balance on account i is x_i. On bank account i, you have a credit interest rate of $\alpha_i > 0$ and a debit interest rate of $\beta_i > \alpha_i$. This means that if $x_i \geq 0$, then you get $\alpha_i x_i$ from the bank. If $x_i < 0$, then you get $\beta_i x_i < 0$ from the bank, which in fact means that you have to pay $\beta_i \cdot |x_i|$.

For compactness, let $f_i : \mathbb{R} \to \mathbb{R}$ be given by

$$f_i(x_i) = \begin{cases} \alpha_i x_i & \text{if } x_i \geq 0 \text{ and} \\ \beta_i x_i & \text{if } x_i < 0. \end{cases}$$

Your goal is to maximize your profit, i.e., to maximize $\sum_{i=1}^{n} f_i(x_i)$.

The balances must satisfy the linear constraints

$$Ax \geq b$$

for some $A \in \mathbb{R}^{m \times n}$ and some $b \in \mathbb{R}^m$ and with $x = (x_1, \dots, x_n)^{\mathrm{T}}$.

Note that you do NOT know any upper bound M such that $|x_i| \leq M$ for all i.

Rewrite this optimization problem as a linear optimization problem without any integer variables.

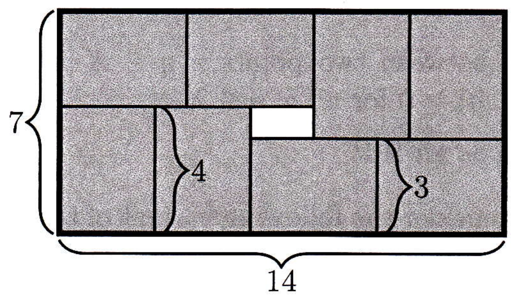
(b) *(4 points)* Now we consider the following variant of the problem: we remove the assumption that $\beta_i > \alpha_i$ for all i, and we add the assumption that $|x_i| \leq M$ for some large, fixed number M and all i.

Rewrite this optimization problem as a mixed integer linear problem.

3. Pallet Loading

Consider a (rectangular) pallet of length A mm and width B mm. We want to load this pallet with as many rectangular boxes as possible. All boxes have a length of a mm and a width of b mm. The height of the boxes is known as well, but not important for this problem since we keep the same side up during loading. Hence we only need to construct one layer of boxes with as many boxes as possible. The boxes are only put parallel to the sides of the pallet. (This means that they can be rotated by 90 degrees.) All dimensions A, B, a, b are integers.

Below you see an example of an optimal packing for the case $A = 7$, $B = 14$, $a = 3$, and $b = 4$.

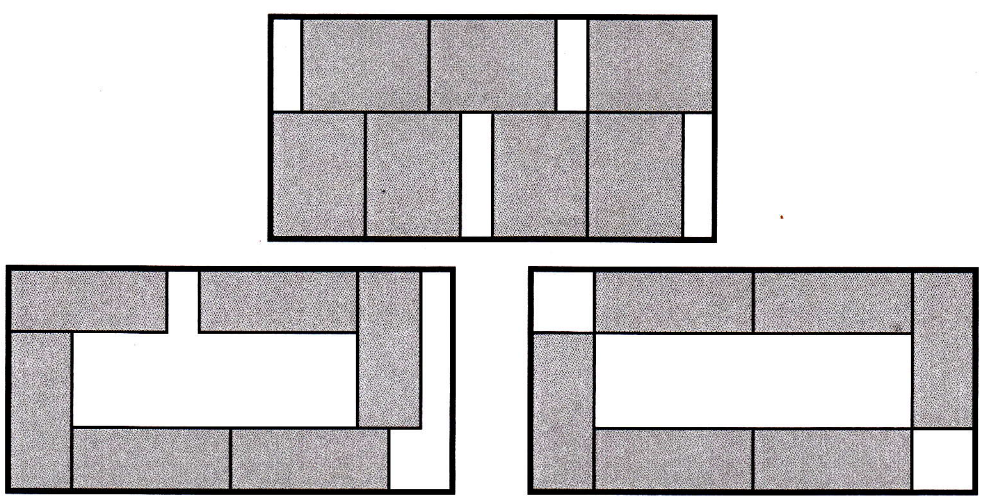


(a) *(8 points)* Model the pallet loading problem as an integer linear program.

(b) *(8 points)* For unknown reasons, hidden free space ("wholes") as in the diagram above is illegal. If there is free space on the pallet, then it must not be fully surrounded by boxes.

See below for examples: The top-most figure below shows an optimal packing for the example described above that satisfies this additional requirement. At the bottom left is a legal (but not optimal) packing for the case $a = 2$, $b = 5$, $A = 7$, $B = 14$. At the bottom right is an illegal packing with the same parameters.

Change your model to take this modification into account.



4. Clustering

We want to cluster a set X of points. More precisely, given a number k, we want to partition the points in X into subsets $C_1, \dots, C_k$. This means that $\bigcup_{i=1}^k C_i = X$ and $C_i \cap C_j = \emptyset$ for all $i \neq j$.

(a) *(7 points)* Our goal is to minimize the sum of the distances of points within the same cluster, namely the function

$$\sum_{i=1}^{k} \sum_{x,y \in C_i} d(x,y),$$

where the distance between two points $x, y \in X$ is given by $d(x,y)$. We have $d(x,x) = 0$ and $d(x,y) \geq 0$ for all $x, y \in X$.

Model this problem as an ILP.

(b) *(7 points)* Now we consider the following variant of the problem: The points X are on the real line, i.e., $X \subseteq [0, M]$ for some fixed large number M. In addition to the clusters $C_1, \dots, C_k$, we are looking for k points $z_1, \dots, z_k$ (called *representatives*) such that

$$\sum_{i=1}^{k} \sum_{x \in C_i} |x - z_i|$$

is minimized. This means that we want to minimize the sum over all clusters of the distances of the points in the cluster to the respective representative.

Model this problem as a MIP.