Exam Optimization Modeling (191581420)

Friday, April 21, 2017, 8:45 – 11:45

- Use of calculators, mobile phones, etc. is not allowed!
- This exam consists of four problems. Start a new page for every problem.
- Total number of points: 50. Distribution of points according to the following table.

1a: 6	2a: 6	3a: 4	4a: 3	5a: 2
1b: 2	2b: 2	3b: 6	4b: 3	5b: 2
		3c: 4	4c: 2	5c: 2
			4d: 2	5d: 2

1. Facility Location

We want to solve the following variant of the facility location problem: An instance is described by an undirected graph $G = (V, E)$ with edge costs $c : E \to [0, \infty)$. Every vertex is a client. Every vertex is a possible location of a facility. Furthermore, we are given a parameter $k \in \mathbb{N}$.

If $v \in V$ is an open facility, client $u \in V$ can be connected to v if there is an edge $e = \{u, v\}$ connecting u and v or if $u = v$. The costs of connecting u to v for $u \neq v$ are $c(e)$. If v is an open facility, then v can be connected to itself with cost 0.

The goal is to open at most k facilities and to connect each client to exactly one facility with minimum total costs.

(a) *(6 points)* Build a MIP model for this problem.

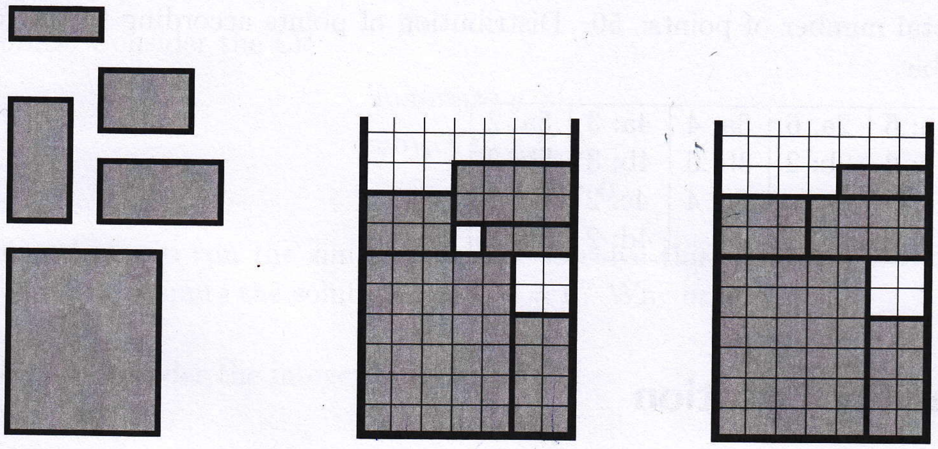
(b) *(2 points)* Now we have the following additional requirement: A facility can only be opened at some $v \in V$ if it serves at least m_v^- clients (excluding itself). Furthermore, it can serve at most m_v^+ clients (excluding itself).

Modify your model of Part (a) accordingly.

2. Tetris

In a simplified offline-version of Tetris, you are given a number of rectangular blocks, and your goal is to stack them in such a way that you use as few lines as possible. (Note that, different from real Tetris, there is no ordering of blocks, i.e., there is nothing like "block a comes after block b and, thus, a cannot be underneath b".)

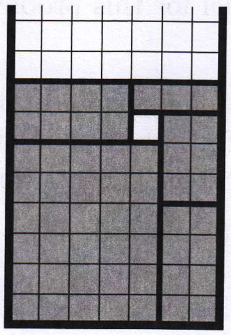
The following graph shows five blocks on the left-hand side and two solutions of a playing field of width $L = 7$ and these five blocks. Both solutions are optimal, as they both need nine lines, and it is impossible to pack the blocks with only eight lines. In this setting, rotation of blocks is not allowed.



(a) *(6 points)* Model the problem described above as an integer linear program. Rotation of blocks is not allowed.

(b) *(2 points)* Now also rotations of blocks are allowed. Describe how you can adapt your model to this variant.

The following graph shows an optimal solution for this setting that requires only eight rows.



3. Knapsack and Column Generation

We consider the binary knapsack problem: we are given items $1, \ldots, n$ and a weight bound B. Item i has a weight of $w_i > 0$ and yields a profit of $p_i > 0$. The goal is to find a subset of the items that weighs at most B and maximizes the profit among all such subsets.

Formulated as an IP, we have

$$\begin{aligned} \text{maximize} \; & \sum_{i=1}^{n} p_i x_i \\ \text{subject to} \; & \sum_{i=1}^{n} w_i x_i \leq B \text{ and } x_1, \ldots, x_n \in \{0,1\}. \end{aligned}$$

We consider the relaxation

$$\begin{aligned} \text{maximize} \; & \sum_{i=1}^{n} p_i x_i \\ \text{subject to} \; & \sum_{i=1}^{n} w_i x_i \leq B, \\ & x_i \leq 1 \text{ for all } i \in [n], \text{ and} \\ & x_1, \ldots, x_n \geq 0. \end{aligned} \tag{K}$$

To avoid special cases and a few technical difficulties, we make the following assumptions:

- We assume that $w_i \leq B$ for all $i \in [n]$ and $\sum_{i=1}^{n} w_i > B$.
- We assume that the items are sorted by decreasing profit-to-weight ratio, and that there are no two items of the same profit-to-weight ratio. This means that

$$\frac{p_1}{w_1} > \frac{p_2}{w_2} > \frac{p_3}{w_3} > \ldots > \frac{p_n}{w_n}.$$

- We assume that no subset of the items has weight exactly B. This means that

$$\sum_{i \in S} w_i \neq B$$

  for all $S \subseteq [n]$. This assumption implies that (K) has a fractional variable in every optimal solution, even if we restrict the instances to a subset of the items.

(a) *(4 points)* Assume that we are given a set $S \subseteq [n]$. Assume further that $\sum_{i \in S} w_i > B$ (this is only to avoid trivial cases – otherwise, we can simply take all items in S).

Give an optimal solution of the relaxation (K) restricted to variables x_i with $i \in S$.

(b) *(6 points)* Write down the dual of (K) and give the dual solution corresponding to your solution of (a).

Hint: Complementary slackness.

(c) *(4 points)* Given your optimal solution from Part (b), what are the reduced costs of the variables x_i with $i \in [n]$?

When should you add a new variable by column generation, or when does the process stop with an optimal solution of (K)? If a variable should be added, which one?

4. Modeling Tricks

(a) *(3 points)* Assume that you have fixed numbers $\ell_1 < u_1 < \ell_2 < u_2$. You have a variable x that should only assume a value in the interval $[\ell_1, u_1]$ or in the interval $[\ell_2, u_2]$.

Formulate this behavior using only linear constraints and appropriate additional (integer) variables.

(b) *(3 points)* Let $k \in \mathbb{N}$ be a fixed natural number, and let $a_1, \ldots, a_k \in \mathbb{R}$ be k fixed real numbers.

Assume that we have an unrestricted variable y and an integer variable x that assumes values in $\{1, 2, \ldots, k\}$.

Model the constraint

$$y = a_x.$$

using only linear constraints. You can use additional variables.

(c) *(2 points)* You have integer variables $x_1, \ldots, x_n$ with range $\{0, \ldots, M\}$ for some $M \in \mathbb{N}$. You also have a fixed integer $b \in \{0, \ldots, M\}$.

Model the constraint $y = |\{i \mid x_i \geq b\}|$ using linear constraints. This means that y should be equal to the number of variables that assume a value of at least b. You can use additional variables.

(d) *(2 points)* You have two unrestricted variables x and y and a fixed number b. Model the constraint $|x - y| = b$ using linear constraints. You can use additional variables.

5. Miscellaneous Questions

(a) *(2 points)* Prove or disprove the following statement: For *every* matrix $A \in \mathbb{Z}^{n \times d}$ with $\operatorname{rank}(A) = n$, there *exists* a vector $b \in \mathbb{Z}^n$ with $b \neq 0$ such that the LP

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}} x \\ \text{subject to } & Ax = b, \\ & x \geq 0 \end{aligned}$$

has only integral basic feasible solutions.

Hint: Cramer's rule.

(b) *(2 points)* Consider the LP

$$\begin{aligned} \text{minimize } & y \\ \text{subject to } & x \leq 2, \\ & x, y \geq 0. \end{aligned}$$

Assume that you run the simplex method to solve this problem. Is it possible that simplex outputs the solution $x = 1$, $y = 0$? Why or why not?

(c) *(2 points)* Consider the integer program

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}} x \\ \text{subject to } & Ax = b, \\ & x \in \mathbb{N}^d. \end{aligned}$$

Let z_{IP} be the objective value of an optimum solution, and let z_{LP} be the objective value of the relaxation of this IP, i.e., when we replace $x \in \mathbb{N}^d$ by $x \geq 0$. (You can assume that both z_{IP} and z_{LP} exist, i.e., both LP and IP are neither unbounded nor infeasible.)

Which of the following cases can occur?

(i) $z_{\mathrm{IP}} < z_{\mathrm{LP}}$?

(ii) $z_{\mathrm{IP}} = z_{\mathrm{LP}}$?

(iii) $z_{\mathrm{IP}} > z_{\mathrm{LP}}$?

(d) *(2 points)* Consider again the IP of Question (c). Assume that $b \in \mathbb{Z}^n$ and that A is a totally unimodular.

Which of the following cases can occur?

(i) $z_{\mathrm{IP}} < z_{\mathrm{LP}}$?

(ii) $z_{\mathrm{IP}} = z_{\mathrm{LP}}$?

(iii) $z_{\mathrm{IP}} > z_{\mathrm{LP}}$?