Exam for LNMB/Mastermath Course on Scheduling
13 May 2013

This exam consists of:

- **2 pages.**
- **5 questions.**
- You can obtain a total of 50 points. Your exam grade will be the points you obtained divided by 5.

When a proof is asked, please provide a mathematically sound proof, short but precise. Unless stated otherwise, you are always expected to (briefly) explain your answer.

In case the objective function is not explicitely specified, it is supposed to be a regular objective function, unless stated otherwise.

Exercise 1 (10 points)

Show that the problem $1 \mid r_j \mid L_{\max}$ is strongly NP-hard by a reduction from Bin Packing.

A definition of the Bin Packing-problem is given at the end of this exam.

Exercise 2 (10 points)

Consider the problem $1 \mid pmtn \mid \sum_j f_j(C_j)$.

(a) Suppose that all f_j are non-decreasing functions, i.e., the objective function is regular. Show that there exists an optimal schedule that does not preempt any job.

(b) Is this also true for arbitrary functions f_j? (the objective function does not need to be regular in this case.)

Exercise 3 (10 points)

Consider the problem $Q \mid \mid C_{\max}$. Given a schedule σ, let $J_i(\sigma)$ be the set of jobs scheduled on machine i and let $L_i(\sigma) = \sum_{j \in J_i} p_j/s_i$ be the load of machine i. A machine c is called a critical machine if $L_c(\sigma) = C_{\max}(\sigma)$.

Consider a schedule σ, such that for any critical machine c, any job $j \in J_c$ and any machine $i \neq c$, we have that $L_i + p_j/s_i \geq C_{\max}(\sigma)$. That is, moving a job from a critical machine to another machine does not improve the schedule.

The goal of this exercise is to show that $C_{\max}(\sigma) \leq \frac{1}{2}(1 + \sqrt{4m-3})C^*_{\max}$, where $C^*_{\max}$ is the optimal makespan.

Assume w.l.o.g. that $s_1 \geq s_2 \geq \ldots \geq s_m$ and that $C^*_{\max} = 1$.

(a) Let $\ell \in J_c(\sigma)$. Show that

$$C_{\max}(\sigma) \leq \frac{\sum_j p_j}{\sum_i s_i} + (m-1)\frac{p_\ell}{\sum_i s_i}.$$

(b) Show that

$$C_{\max}(\sigma) \le \frac{\sum_j p_j}{s_1} \le \frac{\sum_i s_i}{s_1}.$$

(c) Note that $C^*_{\max} \ge p_\ell / s_1$ for any job ℓ. By assumption that $C^*_{\max} = 1$, we therefore know that $p_\ell \le s_1$. Thus from (a) and (b) it follows that

$$C_{\max}(\sigma) \le \min\left(1 + (m-1)\frac{s_1}{\sum_i s_i}, \frac{\sum_i s_i}{s_1}\right)$$

Show that

$$C_{\max}(\sigma) \le \frac{1+\sqrt{4m-3}}{2} \cdot C^*_{\max}.$$

Exercise 4 (10 points)
Consider the permutation flow shop problem, F | *permutation* | $C_{\max}$.

(a) Show that a non-delay algorithm that uses an arbitrary permutation yields a schedule with makespan at most m times the optimal makespan.

(b) Show that this analysis is tight. That is, provide an instance such that the worst permutation yields a makespan equal to m times the optimal makespan.

Exercise 5 (10 points)
Consider the stochastic scheduling problem P$||\mathbb{E}[C_{\max}]$ with $n = 3$ jobs which are independently distributed according to the following distributions:

- Job 1 has deterministic size $p_1 = 2$.
- $\Pr[P_2 = 1] = \Pr[P_2 = 2] = \Pr[P_2 = 3] = 1/3$
- Job 3 has processing time from a uniform distribution over $[1, 3]$.

(a) Give an optimal policy for $m = 1$. $\mathbb{E}[C_1 + C_2 + C_3]$

(b) Give an optimal policy for $m = 2$.

(c) Give an optimal policy for $m = 3$.

For all three situations, determine the expected makespan of an optimal policy.

Definition 1 (Bin Packing) *In the Bin Packing problem, we are given q items of sizes $a_1, \ldots, a_q$ and bins of capacity B. Furthermore, we are given a bound K on the number of bins used. Is it possible to pack the items in at most K bins, such that each item is in exactly one of the bins, and the total size of the items packed in one bin does not exceed the bin capacity B?*

Bin Packing is known to be strongly NP-hard.
You may assume that $0 < a_i \le B$ and that all values are integral.