Course 19.155120.0 “Scientific Computing”
test T_2

April 25, 2012, 13:45–14:05

Your name: __________________________________

Your student number: __________________________________

Space for your drafts (will not be checked)

Question 1 (30 points) Let $A \in \mathbb{R}^{n \times n}$, $r \in \mathbb{R}^n$ and $m \ll n$ be given. What can you say about the following algorithm and the matrices $V_{m+1} \in \mathbb{R}^{n \times (m+1)}$ and $B \in \mathbb{R}^{(m+1) \times m}$ produced by it? Here, v_j denotes column j of V_{m+1}.

define zero matrices $B \in \mathbb{R}^{(m+1) \times m}$ and $V_{m+1} \in \mathbb{R}^{n \times (m+1)}$
$\beta := \|r\|_2$, $v_1 := r/\beta$
for $j = 1, 2, \ldots, m$ do
    $w := Av_j$
    for $i = 1, 2, \ldots, j$ do
        $b_{ij} := (w, v_i)$
    endfor
        $w := w - \sum_{i=1}^{j} b_{ij} v_i$
    $b_{j+1,j} := \|w\|_2$
    if $b_{j+1,j} = 0$ stop
    $v_{j+1} := w / b_{j+1,j}$
endfor

Question 2 (30 points) Assume a linear system $Ax = b$ is being solved for given $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. An iterative method for its solution has a property that its residual r_k after k iterations satisfies $r_k = P_k(A) r_0$, where $r_0 = b - Ax_0$ is the initial residual, x_0 is the initial guess and P_k is a polynomial of degree k. Is this iterative method a Krylov subspace method? Explain why or why not.

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Question 3 (20 points) Assume m steps of the Arnoldi process for a matrix $A \in \mathbb{R}^{n\times n}$ and vector $r \in \mathbb{R}^n$ produced the matrices $V_{m+1} \in \mathbb{R}^{n\times(m+1)}$ and $\underline{H}_m \in \mathbb{R}^{(m+1)\times m}$, $m < n$ Let H_m be formed by the first m rows of $\underline{H}_m$.

(a) Give a definition of the Rayleigh quotients of A. Prove that the Rayleigh quotients of H_m form a subset of the Rayleigh quotients of A.

(b) Assume now that $A = I+S$ where I is the identity matrix and $S = -S^*$. Prove that the Rayleigh quotients of H_m lie in the complex plane on the line $\{z \in \mathbb{C} \,|\, z = 1 + iy,\ y \in \mathbb{R}, i^2 = -1\}$.

Question 4 (20 points) Let $A \in \mathbb{R}^{n\times n}$, $g(t) : \mathbb{R} \leftarrow \mathbb{R}^n$ and $w^0 \in \mathbb{R}^n$ be given. The BDF2 scheme for the time integration of the IVP $w'(t) = -Aw(t) + g(t)$, $w(0) = w^0$ reads:

$$\frac{3}{2}w^{k+2} - 2w^{k+1} + \frac{1}{2}w^k = -\tau A w^{k+2} + g^{k+2}, \quad k \geqslant 0,$$

where w^k is the approximate solution at time $t_k = k\tau$, $w^k \approx w(t_k)$, $\tau > 0$ is the time step size and $g^k = g(t_k)$. Assume also that w^1 is known.

(a) Rewrite BDF2 in the form of a linear system that has to be solved at every time step.

(b) Assume $A = A^T$. Write this linear system in a preconditioned form, such that an application of the MINRES iterative solver would be possible.