Final exam for the course 191551200 "Scientific Computing"
June 26, 2017, 08:45–10:45

It is not allowed to use any electronic equipment or books.
All the answers must be clearly and fully motivated.

1. (a) (5 p) Give a definition of the Schur decomposition of a matrix $A \in \mathbb{C}^{n\times n}$.

   (b) (10 p) A matrix $A \in \mathbb{R}^{3\times 3}$, $A = A^T$, has eigenvectors $v_{1,2,3}$ and corresponding eigenvalues $\lambda_{1,2,3}$ defined as

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -13 \\ 2 \\ 3 \end{bmatrix}, \qquad \lambda_1 = -1, \ \lambda_2 = -2, \ \lambda_3 = -3.$$

   Provide a Schur decomposition of the matrix A. Hint: what can be said on the eigenvectors of A if $A = A^T$? Do not compute A.

   (c) (5p) Based on your answer to the previous question, provide another Schur decomposition of the same matrix A.

2. (5p) For a matrix $A \in \mathbb{R}^{n\times n}$, give a definition an LU factorization with partial pivoting. Does it exist for all $A \in \mathbb{R}^{n\times n}$?

3. (5p) Is it advisable to use a stable algorithm for solution of an ill-posed problem? Why or why not?

4. A linear system $Ax = b$ has to be solved for given nonsingular $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. For a nonsingular $M \in \mathbb{R}^{n\times n}$ and $N = M - A$, consider an iterative method

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b. \tag{1}$$

   (a) (5p) Show that if (1) converges, it converges to the solution $x = A^{-1}b$ of the given linear system.

   (b) (10p) Formulate (derive) and prove a convergence condition for the iterative method (1).

5. For a given $A \in \mathbb{R}^{n\times n}$, an eigenvalue problem $Ax = \lambda x$ has to be solved by the Arnoldi method.

   (a) (5p) Provide the Arnoldi relation for the familiar matrices V_{k+1}, $\underline{H}_k$ and their submatrices.

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(b) (10p) Assume we carried out k steps of the Arnoldi process and find approximate eigenpairs using the Arnoldi method and the Ritz values and vectors. Define the residual $\tilde{r}$ of an approximate eigenpair $(\tilde{\lambda}, \tilde{x})$ and show that $\tilde{r}$ is parallel to the last column of V_{k+1}.

6. (7p) Assume a nonlinear system of equations $F(x)$ has to be solved, where $F : \mathbb{R}^n \to \mathbb{R}^n$ and $F(x)$ can be written as

$$F(x) = Mx - R(x).$$

Here M is a nonsingular matrix and $R(x)$ is defined as the difference $Mx - F(x)$. Formulate a fixed point iteration for solving $F(x) = 0$ involving M^{-1} and $R(x)$. Do we need to compute M^{-1}?

7. (8p) The nonlinear system as defined in the previous question arises at each step of the implicit Euler method applied to solve an ODE system $y' = f(y)$, $f : \mathbb{R}^n \to \mathbb{R}^n$. Specify F in terms of f, time step size $\tau > 0$ and possibly some other values.

8. (5p) Let f be a function $f : \mathbb{R}^n \to \mathbb{R}$. Formulate the Newton method for optimization to solve $f \to \min$.

9. (10p) For given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{k \times k}$, $C \in \mathbb{R}^{n \times k}$, we solve the Sylvester equation $AX - XB = C$. Formulate an algorithm for solving the problem if B is lower triangular. Give an estimate for the computational costs of your algorithm.

The grade is determined as $G = (10 + p)/10$, where p is the total number of points earned.