Final exam for the course 191551200 "Scientific Computing"
Resit. July 5, 2017, 08:45–10:45

It is not allowed to use any electronic equipment or books.
All the answers must be clearly and fully motivated.

1. (a) (5 p) Give a definition of the singular value decomposition (SVD) of a matrix $A \in \mathbb{C}^{m \times n}$. Does it exist for all $A \in \mathbb{C}^{m \times n}$?
   (b) (10 p) Using the SVD, establish and prove a relation between the singular values of a matrix $A \in \mathbb{R}^{m \times n}$ and the eigenvalues of $A^T A$.

2. (5p) For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, give a definition of the Cholesky factorization. Does it exist for all symmetric $A \in \mathbb{R}^{n \times n}$?

3. (5p) When solving a problem on a computer by a numerical algorithm, we can expect three kinds of errors. Which ones?

4. A linear system $Ax = b$ has to be solved for given nonsingular $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.
   (a) (5p) For a nonsingular $M \in \mathbb{R}^{n \times n}$, write down the preconditioned linear system $\widetilde{A}\widetilde{x} = \widetilde{b}$, where the preconditioner M is applied from the right. Specify $\widetilde{A}$, $\widetilde{x}$, $\widetilde{b}$ in terms of A, x, b, M.
   (b) (10p) Rewrite iterative method
   $$x_{k+1} = x_k + M^{-1}(b - Ax_k) \tag{1}$$
   in terms of $\widetilde{A}$, $\widetilde{x}_k$ and $\widetilde{b}$. You get an iterative scheme for solving the preconditioned system $\widetilde{A}\widetilde{x} = \widetilde{b}$. How is this scheme called?
   (c) (5p) Which kind of convergence, linear or superlinear, would you expect for method (1)? Why?

5. (a) (5p) Let $\Lambda(A)$ denote the spectrum of $A \in \mathbb{R}^{n \times n}$. Assume there exist matrices $V \in \mathbb{R}^{n \times k}$ and $H \in \mathbb{R}^{k \times k}$, $k < n$, such that $AV = VH$. What can be said about $\Lambda(A)$ and $\Lambda(H)$? No proof is required. How is the column space of V called in this case?
   (b) (10p) The eigenvalue problem $Ax = \lambda x$ is solved by the Arnoldi method. After k steps of the Arnoldi process are carried out, it turns out that in the familiar matrix $\underline{H}_k$ we have $|h_{k+1,k}| < 2 \cdot 10^{-16}$. Prove that all the eigenvalues of H_k are good approximations to some of the eigenvalues of A.

6. (7p) A nonlinear system of equations $F(x) = 0$ has to be solved for $F : \mathbb{R}^n \to \mathbb{R}^n$. Formulate an inexact Newton iteration method for solving $F(x) = 0$.

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7. (8p) The nonlinear system $F(x) = 0$ as defined in the previous question arises at each step of the implicit trapezoidal rule method applied to solve an ODE system $y' = f(y)$, $f : \mathbb{R}^n \to \mathbb{R}^n$. Specify F in terms of f, time step size $\tau > 0$ and possibly some other values.

8. (5p) Let f be a function $f : \mathbb{R}^n \to \mathbb{R}$. Formulate the steepest descent method to solve $f \to \min$.

9. (10p) Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{k\times k}$. Finish and prove the relation $(A \otimes B)^T = \dots$.

The grade is determined as $G = (10 + p)/10$, where p is the total number of points earned.