Department of Applied Mathematics July 1, 2020, 08:45–11:45
Faculty of Electrical Engineering, Mathematics and Computer Sciences
Dr. K. Smetana, Dr. M. Walter

Resit of the Exam for the Course

Scientific Computing (code: 191551200)

2019/2020

The exam is held based on the assumption that you will do it yourself, without the help from other people. You are only allowed to use

- The lecture notes (including your own annotations to the lectures notes (that means that if you have written something in the pdf or printed version)).
- Your own notes you took during the course.
- All slides and videos available on the Canvas page of the course Scientific Computing 2019/2020 (course code 191551200) at the University of Twente.
- The book Quarteroni, Alfio, Sacco, Riccardo, Saleri, Fausto. Numerical Mathematics, Second Edition, Springer, Berlin Heidelberg New York, 2007.
- The exercise sets and your solutions.
- The practical assignments and your solutions.
- A ruler.

Please read the following paragraphs carefully, and copy the text below in italics to your answer sheet. To find more information, please consult `https://canvas.utwente.nl/courses/4925/discussion_topics/60730`. By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.
Text to be copied: *I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*

Sign this statement, and write below your signature your name, student number and study program.

Students have to submit their solutions, a picture of their student card, and, if applicable, the card allowing extra time as one single pdf file. Other formats than pdf will not be accepted. Please make sure that the scan is clear and good to read. What we cannot read, we cannot grade.

Students will have 25 minutes after the exam is finished to produce this single pdf file and upload it to Canvas; hence till 12:10 for regular students and 12:55 for students with permission for extra time. After this, the site will be closed and no answers will be accepted.

In addition, students have to upload a scan of their scrap paper to Canvas as an additional pdf file. Students will have 30 minutes to complete that task after the respective deadline for uploading their pdf containing their solutions; hence till 12:40 for regular students and 13:25 for students with permission for extra time.

The grades obtained from the exam will not automatically be the officially registered, definite grades. Instead, the grades first enter an intermediate stage of "crowned grades", before becoming "definite grades".

In case we observe significant irregularities, this will be reported to the Examination Board and at least the grade might not be turned into a "definite grade".

In case a picture of the student card, a picture of the card allowing extra time, or the signed integrity statement is missing, this will be reported to the Examination Board and the grade might not be turned into a "definite grade".

Please indicate clearly all the questions you solve and avoid writing your solution on different, disjoint pages.

All answers must be motivated and clearly formulated. Explain each step in your solution. Your solutions should make very clear to us that you understand all of the steps and the logic behind the steps.

The total number of points is 54 points. In order to get the grade 5.5 you will need at most 27 points. This exam has 8 tasks.

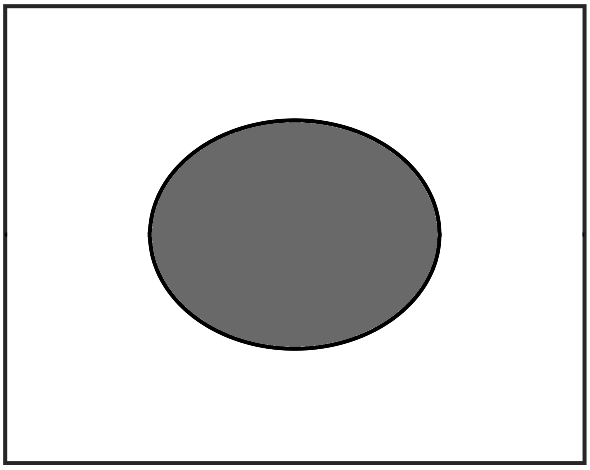
Task 1 (LU decomposition) (4 + 4 points)

We have given the following matrix A:

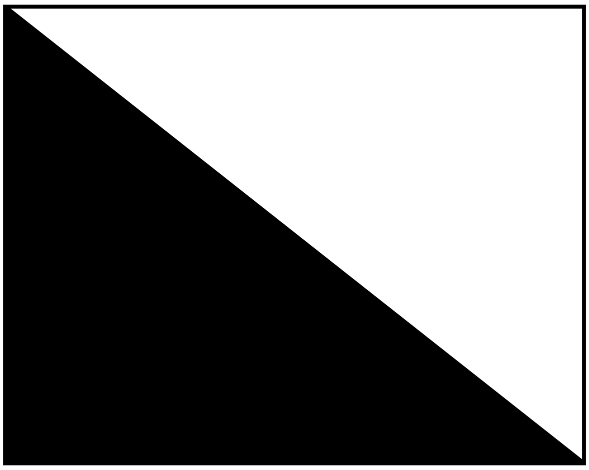
$$A := \begin{pmatrix} a & 0 & 2 \\ 4 & b & 5 \\ 3 & 4 & c \end{pmatrix} \qquad \text{with } a, b, c \in \mathbb{R}. \tag{1}$$

a) Compute the LU decomposition of A defined in (1) <u>without</u> pivoting, where L has to be a unit lower triangular matrix. Check whether your LU decomposition satisfies indeed $A = LU$. For this subtask you can assume that a, b, c are chosen such that a unique LU decomposition exists.

b) Give conditions for a, b, c that are sufficient for the existence of a unique LU decomposition with a unit lower triangular matrix L and an upper triangular matrix U such that $A = LU$ for A defined in (1). Give also a brief explanation.



(a) Picture 2: 1826×1442 pixels



(b) Picture 3: 1830×1449 pixels

Figure 1: Different pictures for task 2.

Task 2 (Singular value decomposition) (5 points)
For $A \in \mathbb{R}^{m\times n}$ with $m \leq n$ we have given a singular value decomposition $A = U\Sigma V^*$ with $V^*V = I_{n\times n}$, $U^*U = I_{m\times m}$ and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_m) \in \mathbb{R}^{m\times n}$. For r, $1 \leq r < m$ we define an approximation $A_r := U\Sigma_r V^*$ with $\Sigma_r := \operatorname{diag}(\sigma_1, \dots, \sigma_r, 0, \dots, 0) \in \mathbb{R}^{m\times n}$.

The pictures in Fig. 1 can be transformed into a matrix such that a black pixel corresponds to a one, a white pixel to a zero, and a gray pixel to a value between zero and one. The number of pixels of each picture are given in the subcaptions of the pictures, where the first number denotes the number of pixels in horizontal and the second number the number of pixels in vertical direction. The black boundary/frame belongs to the picture. What is the smallest number you expect you can choose for r for each picture in order to have an error $\|A - A_r\|_2$ of about zero up to machine precision? Provide an explanation for your answer. A deviation of about 30 from the correct value of r will be deemed still acceptable; the explanation is more important.

Task 3 (Analysis of weak solutions) (2 points)
Explain why it is important to first analyze the existence and uniqueness of the weak solution before dealing with its approximation say with Finite Elements.

Task 4 (Finite Element approximation) (2 + 3 + 6 points)
Let $I = (0, 1)$ and define the vector space $X := \{v \in H^1(I) \,:\, v(0) = 0\}$ (X is the same as in Chapter 4.1 in the lecture notes) equipped with the H^1-norm. Moreover, let $y \in X$ be the weak solution of the problem, satisfying

$$\int_I y'(x)v'(x)\,dx = \int_I x(2x-1)v(x)\,dx \quad \forall v \in X. \tag{2}$$

Additionally, let $I_h := \{x_0, \dots, x_5\} \subset I$ be a mesh with $x_0 = 0,\ x_5 = 1,\ x_j = j/5$, and define $I_j := (x_{j-1}, x_j)$, $j = 1, \dots, 5$. Finally, let $X_h \subset X$ be the linear Finite Element space

$$X_h := \{\phi_h \in C^0([0,1]) \,|\, \phi_h(0) = 0, \phi_h|_{I_j} \in \mathbb{P}_1\ \forall j = 1, \dots, 5\}$$

and $y_h \in X_h$ the Finite Element approximation of the weak solution $y \in X$ of (2). Assume further that $y \in C^2(I)$.

a) Derive the strong form corresponding to (2).

b) Show that we have $y \in H^2(I)$.

c) Identify the cell I_j for which you expect the largest absolute error in the X-norm. If you can add an additional node, where would you place it?

Task 5 (Power iteration) (9 points)

Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and that

$$Q^T AQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

where λ_i are the real eigenvalues of A, $Q = [q_1, \ldots, q_n]$, and q_i denote the orthonormal eigenvectors corresponding to λ_i. Finally, assume

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|.$$

Let the vectors $z^{(k)}$ be obtained by the power iteration using the Rayleigh quotient to define $\lambda^{(k)}$ (Remark 5.14 in the lecture notes), i.e. let the vectors $z^{(k)}$ be obtained by the algorithm

<u>`Input:`</u> $z^{(0)} \in \mathbb{R}^n$ `with` $\|z^{(0)}\|_2 = 1$

`For` $k = 1, 2, \ldots$ `compute`

$$\begin{aligned} \tilde{z}^{(k)} &= Az^{(k-1)} \\ z^{(k)} &= \tfrac{1}{\|\tilde{z}^{(k)}\|_2}\tilde{z}^{(k)} \\ \lambda^{(k)} &= R(A, z^{(k)}) \ . \end{aligned}$$

Finally, define $\phi_k \in [0, \pi/2]$ by $\cos(\phi_k) = |q_1^T z^{(k)}|$. <u>Task:</u> Prove that if there holds $\cos(\phi_0) \neq 0$, then for $k = 0, 1, 2, \ldots$ we have

$$|\lambda^{(k)} - \lambda_1| \leq \max_{2\leq i\leq n} |\lambda_1 - \lambda_i| \tan(\phi_0)^2 \left|\frac{\lambda_2}{\lambda_1}\right|^{2k}. \tag{3}$$

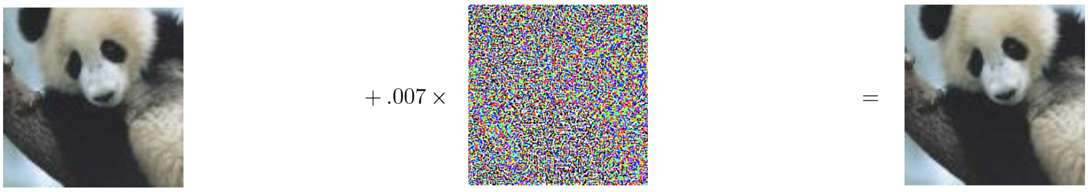
<u>Hint:</u> Use ideas from the proof of Theorem 5.13 in the lecture notes and from the solution to exercise 3 on exercise set 6, which can also be found as Theorem 5.5 on page 190 in the book by Quarteroni, Sacco, and Saleri. Start the proof by rewriting $\lambda^{(k)}$ and finally use that $\tan(\phi_0) = \sin(\phi_0)/\cos(\phi_0)$.

Task 6 (Comparison of algorithms that approximate singular values and singular vectors) (10 points)

Assume that you want to compute approximations of the singular values $\sigma_1, \ldots, \sigma_{20}$ and the corresponding left singular vectors $u_1, \ldots, u_{20}$ of a large sparse matrix $A \in \mathbb{R}^{m\times n}$, $m, n \gg 20$. Which algorithms that we discussed during the course can be recommended to compute such an approximation and why? Comment also briefly why other methods we discussed are not suited for the current task. Compare the key ideas of those (two) algorithms. Which algorithm would you prefer in this setting; consider the convergence behavior/rate and the computational costs in this context.

Task 7 (Neural network) (3 points)

Reflect on the results by a neural network on an image classificiation task as shown in Fig. 2. Here, $\epsilon = 0.007$ corresponds to the magnitude of the smallest bit of an 8 bit image encoding after the neural network's conversion to real numbers. The notation follows Chapter 8 in the lecture notes.



(a) x
"panda"
57.7% confidence

(b) $\text{sign}(\nabla_x \text{Cost}(p))$
"nematode"
8.2% confidence

(c) $x + \epsilon\, \text{sign}(\nabla_x \text{Cost}(p))$
"gibbon"
99.3% confidence

Figure 2: Results of a neural network on an image classificiation task, where $\epsilon = 0.007$ corresponds to the magnitude of the smallest bit of an 8 bit image encoding after the neural network's conversion to real numbers.

Task 8 (Mixed-integer Programming Model) (6 points)

Eni wants to schedule the flour production of her mill for a day. She can produce different flour types $K := \{1, 2, \dots, n\}$, and for each $k \in K$ she has a demand of $d_k^1 \in \mathbb{Z}_{\geq 0}$ at noon (after 720 potential production minutes) and an additional demand of $d_k^2 \in \mathbb{Z}_{\geq 0}$ at midnight (after another 720 potential production minutes). Thus, the total demand of flour $k \in K$ is $d_k^1 + d_k^2$, for which she may decide to produce more than d_k^1 in the morning and less than d_k^2 in the afternoon. The production times are given as $p_k \in \{5, 10, 15, 20, \dots\}$. However, whenever the flour type is changed the mill needs 10 minutes reconfiguration time in which nothing can be produced. At the beginning, no reconfiguration is necessary.

Create a mixed-integer program to help Eni solve this feasibility problem! Briefly explain the meaning of your variables and constraints (one phrase or sentence each).

overview points

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
8	5	2	11	9	10	3	6

Table 1: Total: 54 points