Problem 1

First we observe that the process described here is a renewal process, since the lifetimes of the machines are independent of each other. The arrivals in this process correspond to the arrivals of a new machine or, equivalently, the break down of the current machine.

a) If we condition the number of arrivals $N(t)$ on the first arrival then because the renewal process starts again after the first arrival and does not depend on this arrival we get

$$\mathbb{E}[N(t)|X_1 = x] = \begin{cases} 0, & \text{if } x > t \\ 1 + \mathbb{E}[N(t-x)] & \text{if } x \leq t. \end{cases}$$

Combining this we the law of total probability we get

$$\begin{aligned} m(t) &= \mathbb{E}[N(t)] \\ &= \int_0^\infty \mathbb{E}[N(t)|X_1 = x]\, dF(x) \\ &= \int_0^t 1 + \mathbb{E}[N(t-x)]\, dF(x) \\ &= F(t) + \int_0^t m(t-x) dF(x). \end{aligned}$$

b) Consider a Renewal Process for which the expected cost incurred per unit time is determined according to:

$$\frac{\mathbb{E}[\text{Cost per cycle}]}{\mathbb{E}[\text{Cycle length}]}$$

Now, the length of a cycle is the time period a machine is functioning. Therefore, the expected time of the cycle is μ. The expected cost per cycle is the cost of getting a new machine, c_1 and the maintenance cost per time unit of functionality multiplied by the expected length of the functionality, which is $c_2\mu$. Thus,

$$\frac{\mathbb{E}[\text{Cost per cycle}]}{\mathbb{E}[\text{Cycle length}]} = \frac{c_1 + c_2\mu}{\mu}.$$

c) We will apply the Renewal theorem to $\mathbb{E}[Y(t)]$. For this we need to show that this satisfies the renewal equation. For this we again condition on the first arrival and use the renewal argument. This gives us

$$\mathbb{E}[Y(t)|X_1 = x] = \begin{cases} x - t & \text{if } t \leq x \\ \mathbb{E}[Y(t-x)] & \text{if } t > x. \end{cases}$$

and hence

$$\mathbb{E}[Y(t)] = \int_0^\infty \mathbb{E}[Y(t)|X_1 = x]\, dF(x) = \int_t^\infty (x-t) dF(x) + \int_0^t \mathbb{E}[Y(t-x)]\, dF(x).$$

Therefore, $\mathbb{E}[Y(t)]$ satisfies the renewal equation with $a(t) = \int_t^\infty (x-t)dF(x)$. In order to apply the Renewal Theorem we first need to show that $\int_0^\infty |a(t)|dt < \infty$,

$$\begin{aligned}
\int_0^\infty |a(t)|dt &= \int_0^\infty a(t)dt \\
&= \int_0^\infty \int_t^\infty (x-t)dF(x)dt \\
&= \int_0^\infty \int_0^x (x-t)dtdF(x) \\
&= \int_0^\infty \frac{1}{2}x^2 dF(x) \\
&= \frac{1}{2}(\sigma^2 + \mu^2) < \infty.
\end{aligned}$$

Now by the Renewal Theorem we get

$$\lim_{t\to\infty} \mathbb{E}[Y(t)] = \frac{1}{\mu}\int_0^\infty a(t)dt = \frac{\mu^2+\sigma^2}{2\mu}.$$

d) For all $t \geq 0$, we have

$$\begin{aligned}
\mathbb{E}[Y(t)] &= \mathbb{E}\left[S_{N(t)+1}\right] - t \\
&= \mu\mathbb{E}[N(t)+1] - t \\
&= \mu\mathbb{E}[N(t)] + \mu - t \\
&= \mu m(t) + \mu - t.
\end{aligned}$$

Now, when T is large enough we have $\mathbb{E}[Y_T] \approx \frac{\mu^2+\sigma^2}{2\mu}$. Hence, using the above equation,

$$m(T) \approx \frac{1}{\mu}(T + \frac{\mu^2+\sigma^2}{2\mu}) - 1.$$

Problem 2

a) To show that Z_n is a martingale, we need to check:

i) $\mathbb{E}[|Z_n|] \leq m \leq \infty$, since the state space is bounded from above by m.

ii) First observe that if $Z_n = 0$ or $Z_n = m$ then $p_{ij} = 0$ for $j \neq 0$, $j \neq m$, respectively. Hence, in this case, we get $\mathbb{E}[Z_{n+1}|Z_n] = Z_n$. Now, suppose that $0 < Z_n < m$. Then

$$\begin{aligned}
\mathbb{E}[Z_{n+1}|Z_n] &= \sum_{j=0}^m P_{Z_n j}\, j \\
&= \sum_{j=0}^m \frac{m!}{(m-j)!j!}\left(\frac{Z_n}{m}\right)^j \left(1 - \frac{Z_n}{m}\right)^{m-j} j \\
&= \sum_{j=1}^m \frac{m!}{(m-j)!j!}\left(\frac{Z_n}{m}\right)^{j-1} \left(1 - \frac{Z_n}{m}\right)^{m-j} j\frac{Z_n}{m}
\end{aligned}$$

Take $k = j - 1$.

$$
\begin{aligned}
&= \sum_{k=0}^{m-1} \frac{m!(k+1)}{m(m-k-1)!(k+1)!} \left(\frac{Z_n}{m}\right)^k \left(1 - \frac{Z_n}{m}\right)^{m-k-1} Z_n \\
&= \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-1-k)!k!} \left(\frac{Z_n}{m}\right)^k \left(1 - \frac{Z_n}{m}\right)^{m-k-1} Z_n \\
&= \left(\frac{Z_n}{m} + 1 - \frac{Z_n}{m}\right)^{m-1} Z_n = Z_n
\end{aligned}
$$

b) Define the following stopping time:

$$T = \min_n \{Z_n = 0 \text{ or } Z_n = m\}.$$

Then, since Z_n is a Markov chain with absorbing states 0 and m, it follows that $\mathbb{E}[T] < \infty$. Moreover, $\mathbb{E}[|Z_{n+1} - Z_n||Z_0, ..., Z_n] \leq m$ for all n because the state space is bounded from above by m. Therefore we can apply Corollary 3.1 (p260) from which it follows that

$$\mathbb{E}[Z_T] = \mathbb{E}[Z_0] = z_0. \tag{1}$$

Now denote by v_0 the probability that state 0 is achieved before state m. Then,

$$\mathbb{E}[Z_T] = v_0\, 0 + (1 - v_0)m = (1 - v_0)m.$$

Plugging this into equation (1), we get

$$v_0 = 1 - \frac{z_0}{m}.$$

c) Since Z_n is a martingale, it is also a submartingale. Additionally, using again the fact that the state space is bounded, $\sup_{n \geq 0} \mathbb{E}[|X_n|] \leq m < \infty$. Hence, by the Martingale Convergence Theorem 5.1(a) (p278), there exists a random variable Z such that

$$\mathbb{P}\left(\lim_{n\to\infty} Z_n = Z_\infty\right) = 1.$$

Because Z_n is an absorbing Markov chain with absorbing states 0 and m, we either end in state 0 with probability $v_0 = 1 - \frac{z_0}{m}$ or in state m with probability $\frac{z_0}{m}$. Therefore it follows that

$$\mathbb{P}(Z = 0) = 1 - \frac{z_0}{m} \quad \text{and} \quad \mathbb{P}(Z = m) = \frac{z_0}{m},$$

which completely determines the distribution of Z.

Problem 3

a) First observe that $\mathbb{E}[X_i] = 0$ and $\mathbb{E}\left[X_i^2\right] = 1$ for all i. For the expectation we then have

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[|S_n^2 - n|\right] \leq \mathbb{E}\left[|S_n^2| + n\right] \leq \mathbb{E}\left[n|X_i^2| + n\right] = 2n < \infty.$$

Furthermore,

$$\begin{aligned}
\mathbb{E}[M_{n+1}|M_n] &= \mathbb{E}[M_{n+1}|S_n] \\
&= \mathbb{E}\left[(S_n + X_{n+1})^2 - (n+1)|S_n\right] \\
&= S_n^2 - n + 2S_n\mathbb{E}[X_{n+1}] + \mathbb{E}\left[X_{n+1}^2\right] - 1 \\
&= S_n^2 - n = M_n,
\end{aligned}$$

which completes the proof that M_n is a martingale.

b) Since T bounds the state space, it follows that X_n is a positive recurrent Markov chain, which implies that the hitting of each state from any other state has finite expectation, hence $\mathbb{E}[T] < \infty$. Moreover, since for all $n < T$, $|S_n| < a$ we have

$$\begin{aligned}
&\mathbb{E}[|M_{n+1} - M_n||S_0, \ldots, S_n] \\
&= \mathbb{E}\left[|S_{n+1}^2 - (n+1) - (S_n^2 - n)|S_0, \ldots, S_n\right] \\
&= \mathbb{E}\left[|2X_{n+1}S_n + X_{n+1}^2 - 1||S_0, \ldots, S_n\right] \\
&\leq \mathbb{E}[|2X_{n+1}S_n||S_0, \ldots, S_n] + \mathbb{E}\left[|X_{n+1}|^2|S_0, \ldots, S_n\right] + 1 \\
&= 2\mathbb{E}[|X_{n+1}||S_n||S_0, \ldots, S_n] + 2 \\
&= 2|S_n|\mathbb{E}[|X_{n+1}||S_0, \ldots, S_n] + 2 \\
&= 2(|S_n| + 1) < 2(a+1) < \infty.
\end{aligned}$$

Hence we can apply Corollary 3.1 (p260), from which we get that

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = a^2 - \mathbb{E}[T],$$

which in turn implies that $\mathbb{E}[T] = a^2$.

c) For Y_n to be a martingale, it needs to satisfy the following two conditions:

i) $\mathbb{E}[|Y_n|] \leq \infty$ for all n, and

ii) $\mathbb{E}[Y_{n+1}|Y_n] = Y_n$ for all n.

The first condition follows, for all finite b and c, from the observation that $Y_0 = 1$ and for $n > 1$,

$$\begin{aligned}
\mathbb{E}[|Y_n|] &= \mathbb{E}\left[|e^{bS_n - cn}|\right] \\
&= \mathbb{E}\left[e^{bS_n - cn}\right] \\
&= \mathbb{E}\left[e^{b\sum_{i=1} X_i}e^{-cn}\right] \\
&= \mathbb{E}\left[e^{bX_i}\right]^n e^{-cn} \\
&= e^{-cn}\left(pe^b + qe^{-b}\right)^n < \infty.
\end{aligned}$$

For the second condition we need that

$$\begin{aligned}
Y_n &= \mathbb{E}\left[Y_{n+1}|Y_n\right] \\
&= \mathbb{E}\left[e^{bS_{n+1}-c(n+1)}|Y_n\right] = Y_n e^{-c}\mathbb{E}\left[e^{bX_{n+1}}\right] \\
&= Y_n e^{-c}(e^b p + e^{-b} q).
\end{aligned}$$

Hence $e^{-c}\left(pe^b + qe^{-b}\right) = 1$.

d) Suppose that Y_n is a martingale then, because Y_n is a Markov chain with positive drift ($p > 1/2$), $\mathbb{P}\left(T_1 < \infty\right) = 1$. Moreover,

$$\begin{aligned}
\mathbb{E}\left[\sup_{n\geq 0}|Y_{n\wedge T_1}|\right] &= \mathbb{E}\left[\sup_{n\geq 0}|e^{bS_{n\wedge T_1}-c(n\wedge T_1)}|\right] \\
&= \mathbb{E}\left[\sup_{n\geq 0}|e^{bS_{n\wedge T_1}}||e^{-c(n\wedge T_1)}|\right] \\
&\leq \mathbb{E}\left[\sup_{n\geq 0}|e^{bS_{n\wedge T_1}}|\right].
\end{aligned}$$

By definition of T_1, $S_{n\wedge T_n} < 1$ for all n. Hence if $b \geq 0$, then $bS_{n\wedge T_1} < b$ for all n and it then follows that

$$\mathbb{E}\left[\sup_{n\geq 0}|Y_{n\wedge T_1}|\right] \leq \mathbb{E}\left[\sup_{n\geq 0}|e^{bS_{n\wedge T_1}}|\right] \leq \mathbb{E}\left[e^b\right] = e^b < \infty.$$

We can now apply Theorem 3.1 (p), to get

$$1 = \mathbb{E}\left[Y_0\right] = \mathbb{E}\left[Y_{T_1}\right] = \mathbb{E}\left[e^{b-cT_1}\right] = e^b\mathbb{E}\left[e^{-cT_1}\right],$$

hence,

$$\mathbb{E}\left[e^{-cT_1}\right] = e^{-b}.$$

From c) we know that if Y_n is a martingale, then $e^{-c}(e^b p + e^{-b} q) = 1$. By solving this equation for e^{-b} we can express $\mathbb{E}\left[e^{-cT_1}\right]$ as a function of c.

Take $x = e^{-b}$, then we arrive at the following quadratic equation:

$$qx^2 - e^c x + p = 0,$$

who's solutions are given by

$$x_\pm = \frac{e^c \pm \sqrt{e^{2c} - 4pq}}{2q}.$$

Note that the function $f(p) = 4p(1-p)$ is decreasing for $p > 1/2$ and $f(1/2) = 1$. Hence, since $e^{2c} > 1$ for $c > 0$ it follows that $e^{2c} - 4pq = e^{2c} - 4p(1-p) > 0$, whenever $c > 0$ and $p > 1/2$.

We now will see which of the two solutions $x_\pm$ we need. Since $x = e^{-b}$ and we need $b \geq 0$ we then must have $x \leq 1$. Because $q < 1/2$, there exists a $k > 2$ such that $kq > 1$. If we take $c = \ln(kq) > 0$ then

$$x_+ = \frac{kq + \sqrt{kq - 4pq}}{2q} > 1,$$

hence

$$\mathbb{E}\left[e^{-cT_1}\right] = e^{-b} = x_- = \frac{e^c - \sqrt{e^{2c} - 4pq}}{2q}.$$

Problem 4

a) i) Let $\{X(t), t \geq 0\}$ be a stationary Gaussian process. Then, for all $t \geq 0$, $X(t)$ and $X(0)$ have the same distributions. Hence

$$\mathbb{E}[X(t)] = \mathbb{E}[X(0)] = \mu < \infty \quad \text{for all } t \geq 0.$$

Let $s, t \geq 0$ with $s \leq t$. Then, using the fact that $X(t_1), X(t_2)$ and $X(t_1 + s), X(t_2 + s)$ have the same joint distributions for all $t_1, t_2 \geq 0$, we get

$$\mathbb{E}[X(s)X(t)] = \mathbb{E}[X(0+s)X((t-s)+s)] = \mathbb{E}[X(0)X(t-s)].$$

From this it follows that

$$\mathrm{Cov}(X(s)X(t)) = \mathbb{E}[X(s)X(t)] - \mathbb{E}[X(s)]\,\mathbb{E}[X(t)] = \mathbb{E}[X(0)X(t-s)] - c^2$$

which proves that $\mathrm{Cov}(X(s)X(t))$ only depends on $t - s$.

ii) Let $\{X(t), t \geq 0\}$ be a Gaussian process which satisfies the given properties and take $s, t_1, \ldots, t_n \geq 0$. The joint density function $f(\vec{x})$ of $X(t_1), \ldots, X(t_n)$ of a Gaussian process is given by

$$f(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu})\Sigma(\vec{x} - \vec{\mu})^T\right\}$$

where $\vec{\mu} = (\mathbb{E}[X(t_1)], \ldots, \mathbb{E}[X(t_n)])$, Σ is the covariance matrix of $X(t_1), \ldots, X(t_n)$, i.e. $\Sigma_{ij} = \mathrm{Cov}(X(t_i)X(t_j))$ and $|\Sigma|$ is it's determinant. Because $\mathbb{E}[X(t)] = c$ for all $t \geq 0$ we get that $\vec{\mu}$ is the constant c vector. Moreover, since $\mathrm{Cov}(X(s)X(t))$ depends only on $t - s$ for $s \leq t$ we get

$$\mathrm{Cov}(X(t_i + s)X(t_j + s) = \mathrm{Cov}(X(t_i)X(t_j)) = \Sigma_{ij}.$$

This implies that the covariance matrix $\hat{\Sigma}$ of $X(t_1 + s), \ldots, X(t_n + s)$ equals Σ. Therefore, the joint density function $\hat{f}(\vec{x})$ of $X(t_1 + s), \ldots, X(t_n + s)$ equals $f(\vec{x})$ which proves that $X(t_1), \ldots, X(t_n)$ and $X(t_1 + s), \ldots, X(t_n + s)$ have the same joint distribution.

b) We first establish the identity in the hint.

$$\begin{aligned}
Z(t+s) &= e^{-(t+s)}B(e^{2(t+s)}) \\
&= e^{-(t+s)}\left(B(e^{2(t+s)}) + B(e^{2t}) - B(e^{2t})\right) \\
&= e^{-(t+s)}B(e^{2t}) + e^{-(t+s)}\left(B(e^{2(t+s)}) - B(e^{2t})\right).
\end{aligned}$$

Now since $B(e^{2(t+s)}) - B(e^{2t}) = \mathcal{N}(0, e^{2(t+s)} - e^{2t})$ it follows that

$$\begin{aligned}
e^{-(t+s)}\left(B(e^{2(t+s)}) - B(e^{2t})\right) &= \mathcal{N}(0, e^{-2(t+s)}(e^{2(t+s)} - e^{2t})) \\
&= \mathcal{N}(0, 1 - e^{-2s}) \\
&= \sqrt{1 - e^{-2s}}\,\mathcal{N}(0, 1),
\end{aligned}$$

which proves the required identity.

c) We compute the covariance of $Z(t)$ process as follows:

$$\begin{aligned} Cov(e^{-t}B(e^{2t}), e^{-s}B(e^{2s})) &= e^{-t}e^{-s}Cov(B(e^{2t}), B(e^{2s})) \\ &= e^{-t}e^{-s} \min e^{2t}, e^{2s} = e^{|t-s|} \end{aligned}$$

d) Firstly, we showed in c) that the covariance of the $Z(t)$ process depends only on $t - s$.

Secondly,

$$\mathbb{E}\left[Z(t)\right] = \mathbb{E}\left[e^{-t}B(e^{2t})\right] = 0 < \infty,$$

where in the last equality we used that $B(e^{2t})$ is B.M. with mean 0.

It follows now from a) that $Z(t)$ is a stationary Gaussian process.