191531750 Stochastic Processes
Exam. Date: 30-01-2015, 13:45-16:45

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 6 problems. The total number of points is 36.
Good luck!

1. [2pt] Let $X_1, X_2, \dots$ be independent exponential random variables with parameter λ. Denote $S_0 = 0$, $S_n = X_1 + X_2 + \cdots + X_n$. Use the Poisson process $N(t)$ to find the expression for $P(S_n \le t)$.

2. a) [3pt] Consider a system that alternates between *on* and *off* states. The *on*-times (*off*-times) are independent and identically distributed random variables with a non-arithmetic distribution, mean μ_1 (μ_2) and variance σ_1^2 (σ_2^2). The system initiates at time $t = 0$ with an *on*-period. Let $p(t)$ be the probability that the system is *on* at time t. Use the Renewal Theorem to prove that

$$\lim_{t\to\infty} p(t) = \frac{\mu_1}{\mu_1 + \mu_2}.$$

   b) [2pt] Jobs arrive to a one-server queue according to a Poisson process with rate λ. The service times of customers are independent and identically distributed, with continuous distribution and mean S. If an arriving job finds the system busy, it is blocked and lost for the system. Find the limiting fraction of time that the server is vacant, as time goes to infinity.

   c) [2pt] In a), let a cycle consist of an *on*-time and a subsequent *off*-time. Assume that *on*- and *off*-times in a cycle are independent. Denote by γ_t the excess time, from time t till the start of a new cycle. Write down the expression for $\lim_{t\to\infty} E(\gamma_t)$.

3. Let $Y_1, Y_2, \dots$ be independent identically distributed random variables with mean $\mu > 0$. For fixed $0 < \beta < 1$, let a be the smallest value u for which $\beta E[\max\{u, Y_1\}] \le u$. (Note that a solves the equation $a = \beta E[\max\{a, Y_1\}]$ and that $a > 0$.) Set $f(x) = \max\{a, x\}$. Denote $M_n = \max\{Y_1, Y_2, \dots, Y_n\}$. Let $X_n = \beta^n f(M_n)$.

   a) [3pt] Show that $\{X_n\}$ is a non-negative supermartingale. Clearly motivate every step in the derivation.

   b) [2pt] Show that $E(\beta^T f(M_T)) \le a$ for any stopping time T.

   c) [2pt] Define $T^* = \min\{n \ge 1 : Y_n \ge a\}$. Show that $a = E[\beta^{T^*} M_{T^*}]$ (*Hint:* consider the process $\{X_n\}$ only up to time T^*). Hence, argue that T^* maximizes $E(\beta^T M_T)$.

   d) [2pt] *Example.* Take $\beta = 1/2$, and assume that Y_1 takes values 0 and 2 with equal probability: $P(Y = 0) = 1/2$, $P(Y = 2) = 1/2$. Compute a, find the distribution of T^*, and verify that $a = E[\beta^{T^*} M_{T^*}]$.

4. Consider an urn in which at stage 0 we start with 2 red balls and 1 green ball. At each stage a ball is drawn uniformly at random from the urn. After a ball is drawn it is put back in the urn and another ball of the same color is added.

   a) [2pt] Let Y_n denote the number of red balls at the end of stage n. Prove that for $2 \leq i \leq n+2$, $P(Y_n = i) = 2(i-1)/((n+1)(n+2))$.

   b) [2pt] Let X_n denote the fraction of red balls at the end of stage n. Prove that $\{X_n\}_{n=0}^{\infty}$ is a martingale.

   c) [3pt] Prove that the fraction of red balls is converging with probability one to a random variable X_∞, with $P(X_\infty \leq x) = x^2$.

5. Let $\{X(t), t \geq 0\}$ be a Brownian motion with zero drift and unit variance, starting at zero.

   a) [1pt] Give the definition of Brownian motion with drift μ and variance σ^2, starting at x_0.

   b) [3pt] Derive an expression for $P(\max_{0 \leq u \leq t} X(u) \leq x)$. Express your answer in terms of the probability density function of a zero mean normal distribution, $p(z,t) = (2\pi t)^{-1/2} \exp(-z^2/(2t))$. (*Hint: Use a reflection principle.*)

   c) [2pt] Let $\{Y(t), t \geq 0\}$ be a zero mean Gaussian process with continuous paths and covariance function $\mathrm{Cov}(Y(s), Y(t)) = s(1-t)$, $0 \leq s \leq t \leq 1$. Prove that

$$Z(t) = (1+t)Y\left(\frac{t}{1+t}\right)$$

   is a Brownian motion with zero drift and unit variance, starting at zero.

6. Let $\{X(t), t \geq 0\}$ be a Brownian motion with zero drift and unit variance, starting at zero. Consider two boundaries a and b, $a < 0 < b$. Let T denote the time that $\{X(t), t \geq 0\}$ first hits a or b. Let P_b denote the probability that b is hit before a. We know that $P_b = -a/(b-a)$.

   a) [3pt] Let $Y(t) = X(t)^3 - 3tX(t)$. Show that $\{X(t), t \geq 0\}$ is a martingale w.r.t. $\{X(t), t \geq 0\}$.

   b) [2pt] Assume that the required conditions for applying the martingale stopping theorem hold and find $\mathrm{Cov}(T, X(T))$.

Total: 36 points