153175 Stochastic Processes
Final Exam. Date: 01.02.2007 Time: 13:30-16:30

In all answers: motivate your answer. When derivation is required, you must provide the derivation. It is not allowed to state that answer is by analogy with results in the book of Ross. This exam consists of 4 exercises. Good luck!

1. Consider a continuous time stochastic process $\{X(t), t \geq 0\}$ that can assume only the two states 1 and 2. If the process is currently at state i, it moves to the next state after an exponentially distributed time with mean $1/\lambda_i$ for $i = 1, 2$. The next state is state 1 with probability p_1 and state 2 with probability $p_2 = 1 - p_1$ irrespective of the past of the process.

   a) Use the renewal reward argument to find the long-run fraction of time the process $\{X(t)\}$ is in state i for $i = 1, 2$.

   b) Let $Y(t)$ be the excess time of a renewal process in which the inter-occurrence times have an H_2 distribution with density $p_1\lambda_1 e^{-\lambda_1 t} + p_2\lambda_2 e^{-\lambda_2 t}$. Argue that

   $$\lim_{t\to\infty} \mathbb{P}(Y(t) > x) = \frac{p_1\lambda_2}{p_1\lambda_2 + p_2\lambda_1} e^{-\lambda_1 x} + \frac{p_2\lambda_1}{p_1\lambda_2 + p_2\lambda_1} e^{-\lambda_2 x}.$$

   c) Compare the results in a) and b). Explain the analogy between them.

   d) Let $Y(t)$ be the excess time of a renewal process, in which the inter-occurrence times have a non-lattice distribution F with mean μ. Then

   $$\lim_{t\to\infty} \mathbb{P}(Y(t) > x) = \frac{\int_x^\infty (1 - F(y))\, dy}{\mu}. \tag{1}$$

   Now, use the renewal reward argument to obtain the *long-run fraction of time* when $[Y(t) > x]$. Under which conditions your result holds? Which additional condition on F is needed so that (1) holds? Give a comment.

2. The winnings per unit stake on game n is X_n, where $X_1, X_2, \ldots$ are independent identically distributed random variables with $\mathbb{P}(X_n = 1) = p > 1/2$, $\mathbb{P}(X_n = -1) = 1-p = q$. At game n, a gambler bets an amount $c_n Z_{n-1}$, where Z_{n-1} is the gambler's fortune after $n - 1$ games, and $c_n \in [0, 1]$ is the fraction of fortune that he is willing to bet. The gambler plays a fixed number N of games. The goal is to optimize the 'interest rate', $\mathbb{E}[\log(Z_N/Z_0)]$, where Z_0 is the gambler's initial fortune.

   a) Show that for any strategy $(c_1, c_2, \ldots, c_N)$, the process $\log(Z_n/Z_0) - n\alpha$ is a supermartingale, where

   $$\alpha = p\log(p) + q\log(q) + \log(2).$$

   Hint: Use the fact that $p\log(1 + x) + q\log(1 - x)$ is maximized when $x = p - q$.

   b) Use the Martingale Stopping Theorem for supermartingales to show that $\mathbb{E}[\log(Z_N/Z_0)] \leq N\alpha$. Argue that the Stopping Theorem is applicable.

c) For which strategy, $\log(Z_n/Z_0) - n\alpha$ is a martingale? What is the gambler's optimal strategy and what is the optimal value of $\mathbb{E}[\log(Z_N/Z_0)]$?

3. a) Let $\{S_n,\ n \geq 0\}$ be a simple random walk. Give the distribution $\mathbb{P}(S_n \leq s)$, $n \in \mathbb{N}$, $s \in \mathbb{Z}$.

   Consider an insurance company, in which claims arrive according to a Poisson process with rate λ, claim sizes are exponentially distributed with mean $1/\nu$, and premium is received at constant rate c. Let the initial capital be A, and assume that the company is allowed to borrow money when the cash position becomes negative (rood staan), so that the company cannot be ruined.

   b) Obtain an expression (formula) for the expected profit of the company just after the n-th claim.

4. Let $\{X(t),\ t \geq 0\}$ be Standard Brownian motion. Let $L(t) = \max\{s < t : X(s) = 0\}$, i.e., $L(t)$ is the time of the last zero before t, and let $M(t) = \max_{0\leq s\leq t} X(s), \quad t \geq 0$. Define

$$Y(t) = \exp\{cX(t) - c^2 t/2\}, \quad t \geq 0.$$

   a) Show that $\{Y(t),\ t \geq 0\}$ is a martingale with mean 1.

   b) Is $Y(t)$ Brownian motion? Motivate your answer.

   c) Derive the distribution of $M(t) - X(t)$.

   d) Show that

$$\mathbb{P}(M(t) > a | M(t) = X(t)) = \exp\{-a^2/2t\}, \quad a > 0.$$

   e) Show that the density of $L(t)$ is

$$\mathbb{P}(L(t) = s) = \frac{1}{\pi\sqrt{s(t-s)}}, \quad s < t.$$

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	4(e)	Total
2	2	1	4	4	3	2	2	4	3	1	2	3	3	36