191531750 Stochastic Processes
Exam. Date: 31-01-2014, 13:45-16:45

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 4 problems. The total number of points is 36.
Good luck!

1. Consider a renewal process, where $X_1, X_2, \dots$ are interarrival times with non-arithmetic distribution $F(\cdot)$, finite expectation μ and finite variance σ^2. Denote
$$S_0 = 0, \quad S_n = X_1 + X_2 + \cdots + X_n.$$
Let $M(t)$ be the renewal function for this renewal process.
a) [3pt] Derive $\lim_{t\to\infty} E(\gamma_t)$ using the Renewal Theorem.
b) [2pt] Show that $N(t) + 1$ is a stopping time for the process $\{S_n, n = 0, 1, \dots\}$.
c) [3pt] Use the Martingale Stopping Theorem to prove the identity
$$t + E(\gamma_t) = \mu(M(t) + 1).$$
Hint: Use the martingale $\{S_n - n\mu, n = 0, 1, \dots\}$ and the stopping time $N(t) + 1$.
d) [2pt] Assume that $X_1, X_2, \dots$ are independent identically distributed (i.i.d.) time intervals between subsequent purchases at a web shop that sells products worldwide (thus, the orders arrive around the clock). Let $E(X_i) = 1$ hour, $\mathrm{Var}(X_i) = 2$ hours. Compute the approximated number of purchases in 48 hours.

2. Suppose that $X_1, X_2, \dots$ are i.i.d. random variables such that for some fixed $\lambda > 0$ we have
$$\varphi(\lambda) := E[e^{\lambda X_n}] \le 1.$$
Let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$.
a) [2pt] Fix $l > 0$. Define $Y_n = e^{-\lambda(l - S_n)}$. Show that $\{Y_n, n = 0, 1, \dots\}$ is a non-negative supermartingale.
b) [3pt] Take $T = \min_{n\ge0}\{S_n > l\}$ and use the Optional Stopping Theorem for non-negative supermartingales to prove that
$$P(\sup_{n\ge0} S_n > l) \le e^{-\lambda l}.$$
c) [3pt] Define $Z_n = Y_n(\varphi(\lambda))^{-n}$. Show that $\{Z_n, n = 0, 1, \dots\}$ is a martingale, and that for each $n = 1, 2, \dots$ it holds that
$$P(S_n > l) \le e^{-\lambda l}(\varphi(\lambda))^n.$$

<See other side>

3. Let $\{S_n, n = 0, 1, \dots\}$ denote a simple random walk on the integers, i.e. $S_n = X_1 + X_2 + \cdots + X_n$, where the X_k are i.i.d. with $P(X_n = -1) = P(X_n = 1) = 1/2$. Let a be a positive integer and denote by T the first time that $\{S_n\}$ reaches $-a$, i.e. $T = \inf\{n > 0 : S_n = -a\}$. In this problem you will prove that T is finite with probability one.

   For this purpose let $M_n = S_{T \wedge n}$, where $T \wedge n = \min\{T, n\}$.

   a) [2pt] Prove that $\{M_n, n = 0, 1, \dots\}$ is a martingale w.r.t. $\{X_n, n = 0, 1, \dots\}$.

   b) [2pt] State the martingale convergence theorem; include *convergence with probability one* as well as *convergence in the mean.*

   c) [2pt] Prove that $\{M_n, n = 0, 1, \dots\}$ converges with probability one.

   d) [2pt] Using the result of part c) prove that T is finite with probability one.

4. Put option. We consider the price of a stock $\{S(t), t \geq 0\}$ as given by $S(t) = S(0)\exp(\mu t + \sigma X(t))$, where $\{X(t), t \geq 0\}$ is Brownian motion with zero mean and unit variance starting at 0. Let $\Phi(x) = \int_{-\infty}^{x} 1/\sqrt{2\pi}\exp(-y^2/2)\mathrm{d}y$.

   a) [1pt] Give the definition of Brownian motion.

   b) [3pt] Let $b < a$. Derive an expression for $P(\min_{0 \leq u \leq t} S(u) \leq b | S(0) = a)$ in terms of $\Phi(x)$. (*Hint:* Use the reflection principle.)

   The discounted stock price at time t, in the presence of interest rate r, is given by $\exp(-rt)S(t)$. Consider the European put option that allows to sell stock at time T for price K. The price of this option is the expected value of the option for that value μ for which $\{\exp(-rt)S(t), t \geq 0\}$ is a martingale.

   c) [3pt] Find the value μ for which the discounted stock price $\{\exp(-rt)S(t), t \geq 0\}$ is a martingale. Prove explicitly that it is a martingale.

   d) [3pt] Derive the price of the put option, i.e. compute

   $$E\left[e^{-rT}\max\{K - S(T), 0\}\right]$$

   for the value of μ computed in c). Express your answer in terms of $\Phi(x)$.

Total: 36 points