Solution/Correction standard, Test Mathematics A; September 18, 2015.

1. $A_1 = [-1, 1)$, $A_2 = [-2, \frac{1}{2})$, ..., $A_5 = [-5, \frac{1}{5})$ (correct interpretation of A_k)

   So
   $$\bigcap_{k=1}^{5} A_k = [-1, \frac{1}{5})$$
   and
   $$\bigcup_{k=1}^{5} A_k = [-5, 1).$$
   and so
   $$\overline{\bigcup_{k=1}^{5} A_k} = [1, 5].$$

2. The statement is true on domain $\mathbb{N}$:
   Take $x = 1$, then $\forall y \left[(1^2 + 1)y = (1 + 1)\sqrt{y^2}\right]$, since $y > 0$.

   The statement is false on domain $\mathbb{Z}$:
   Since then the statement must be true for all $y > 0$,
   so $x^2 + 1 = x + 1$, and so $x = 0$ or $x = 1$
   But if $x = 0$ or $x = 1$ then the statement is false for $y < 0$. (if $y < 0$ then $\sqrt{y^2} = -y$)

3. By definition, an integer n is divisible by 6 if it can be written as $n = 6\ell$ for some integer ℓ.

   Basis step for $n = 1$:
   $7^1 - 1 = 6 = 6 \cdot 1$ (take $\ell = 1$).
   So the statement is correct for $n = 1$.

   Induction step:
   Let $k \geq 1$ and suppose that:
   $7^k - 1$ is divisible by 6, so $7^k - 1 = 6\ell$ for some $\ell \in \mathbb{Z}$ (Induction hypothesis: IH)

   We must show that IH implies: $7^{k+1} - 1$ is divisible by 6,
   so we must show that there is an integer $m \in \mathbb{Z}$ such that $7^{k+1} - 1 = 6m$.
   Well: $7^{k+1} - 1 = 7 \cdot 7^k - 1$. Now applying IH ($7^k = 6\ell + 1$) we get:

   $7 \cdot 7^k - 1 = 7 \cdot (6\ell + 1) - 1 = 7 \cdot 6\ell + 7 - 1 = 6 \cdot 7\ell + 6 = 6 \cdot (7\ell + 1).$

   Hence, $7^{k+1} - 1 = 6m$ for $m = 6\ell + 1$. This proves the induction step.
   Now we obtain from the principle of mathematical induction that for all $n \geq 1$:
   $7^n - 1$ is divisible by 6.

4. (a) Choose 5 places out of 10 for the zeros: $\binom{10}{5}$ possibilities.

The other 5 places can be filled with 1's, 2's or 3's: 3^5 possibilities.

So the total number of strings is: $\binom{10}{5}3^5 \qquad (= 17.010).$

(b) The number of zeros is exactly 5, and of the remaining 5 places the number of ones is exactly 3 or exactly 4 or exactly 5 (and the rest of the places have 2's or 3's).
The number of strings corresponding to these situations is:

$\binom{10}{5}\binom{5}{3}2^2, \quad \binom{10}{5}\binom{5}{4}2^1$ and $\binom{10}{5}\binom{5}{5}2^0.$

So the total number of strings is:

$$\binom{10}{5}\binom{5}{3}4 + \binom{10}{5}\binom{5}{4}2 + \binom{10}{5}\binom{5}{5} \qquad (= 6.930).$$