Solution/Correction standard, Test Mathematics B1; October 4, 2013.

1. (a) Several possible solutions:

[i] Method known from secondary school: show that $f'(x) > 0$ for all $x \in \mathbb{R}$.

$f'(x) = \dfrac{2x}{(e^x+1)^2}$ **[0.5 pt]**

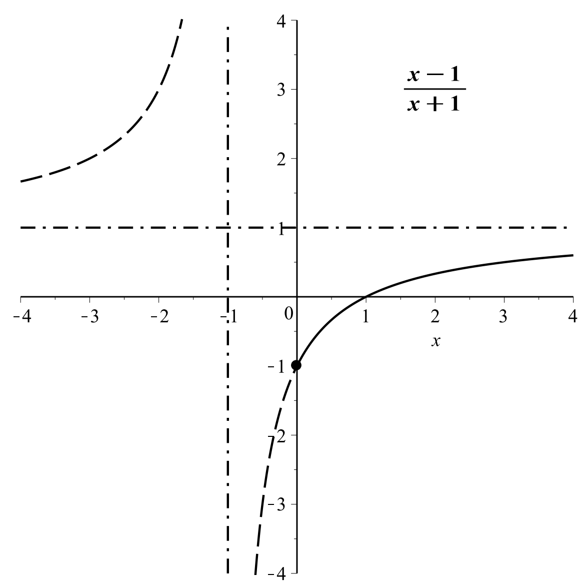
$\dfrac{2x}{(e^x+1)^2} > 0$ for all $x \in \mathbb{R}$ **[0.5 pt]**

[ii] With definition: suppose $f(a) = f(b)$, show that $a = b$: **[0.5 pt]**

$$\begin{aligned} \frac{e^a-1}{e^a+1} &= \frac{e^b-1}{e^b+1} \\ (e^a-1)(e^b+1) &= (e^b-1)(e^a+1) \\ e^{a+b}+e^a-e^b-1 &= e^{a+b}-e^a+e^b-1 \\ e^a-e^b &= -e^a+e^b \\ 2e^a &= 2e^b \\ e^a &= e^b \\ a &= b \end{aligned}$$

Flawless execution of the derivation gives **[0.5 pt]**.

(b)



Write $f(x)$ as $g(e^x)$ where $g(u) = \dfrac{u-1}{u+1}$. The function g maps the interval $(0, \infty)$ onto $(-1, 1)$. The range of f is $(-1, 1)$. This is also the domain of f^{-1}. **[1 pt]**

For the inverse: solve $y = f(x)$:

$$
\begin{aligned}
y &= \frac{e^x - 1}{e^x + 1} \\
y(e^x + 1) &= e^x - 1 \\
ye^x + y &= e^x - 1 \\
ye^x - e^x &= -1 - y \\
(1-y)e^x &= 1 + y \\
e^x &= \frac{1+y}{1-y} \\
x &= \ln\left(\frac{1+y}{1-y}\right) = \ln(1+y) - \ln(1-y).
\end{aligned}
$$ **[1 pt]**

We have $f^{-1} \colon (-1,1) \to \mathbb{R}$ with (replace y by x):

$$f^{-1}(x) = \ln(1+x) - \ln(1-x).$$

(c) Solution 1: differentiate $f^{-1}(x)$:

$$\left(f^{-1}\right)'(x) = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{(1-x)(1+x)} = \frac{2}{1-x^2}.$$ **[0.5 pt]**

From this follows $\left(f^{-1}\right)'(0) = 2$ **[0.5 pt]**

Solution 2: use $\left(f^{-1}\right)'(y) = 1/f'(x)$ where $f(x) = y$:

$$f(0) = 0$$ **[0.5 pt]**

and

$$f'(x) = \frac{2e^x}{\left(e^x + 1\right)^2} \quad \text{therefore} \quad f'(0) = \frac{1}{2}.$$ **[0.5 pt]**

2. Write down the definition of derivative in x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$ **[0.5 pt]**

Calculate this limit for $f(x) = x^2$:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x.
\end{aligned}
$$ **[1.5 pt]**

3. (a) Solve the homogeneous equation using the integrating factor:

$$v(x) = e^{\int 3x^2-1\,dx} = e^{x^3-x}. \qquad \textbf{[0.5 pt]}$$

The general solution then is

$$y(x) = \frac{1}{v(x)} = Ce^{x-x^3}. \qquad \textbf{[0.5 pt]}$$

Use the inital condition to solve C:

$$1 = y(1) = Ce^0 \quad \text{hence} \quad C = 1. \qquad \textbf{[0.5 pt]}$$

Write down the solution:

$$y(x) = e^{x-x^3}. \qquad \textbf{[0.5 pt]}$$

(b) Use the integrating factor $v(x) = e^{x^3-x}$ from (a). Find an antiderivative of $v(x) \cdot x^2 e^x$:

$$\begin{aligned} \int v(x) \cdot x^2 e^x \, dx &= \int e^{x^3-x} x^2 \, e^x \, dx \\ &= \int x^2 \, e^{x^3} \, dx \\ &= \frac{1}{3} e^{x^3}. \end{aligned} \qquad \textbf{[1 pt]}$$

A particular solution is found by dividing the result by v:

$$y(x) = \frac{1}{v(x)} \cdot \frac{1}{3} e^{x^3} = \frac{1}{3} e^{x-x^3} e^{x^3} = \frac{1}{3} e^x. \qquad \textbf{[1 pt]}$$