Course	:	Mathematics β II
Date	:	January 13th 2017
Time	:	13:45-15:45

Please provide motivation for all your answers and calculations. The use of electronic devices is not allowed.

1. Let

$$f(x) = \begin{cases} \dfrac{x^2}{3+\sin(1/x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

(a) Determine $f'(x)$ for $x \neq 0$.

(b) Use the definition of derivative to obtain that $f'(0) = 0$.

(c) Calculate $f'(\frac{1}{2k\pi})$, $k \in \mathbb{N}$.

(d) Is $f'(x)$ continuous in 0?

2. (a) Formulate the Mean Value Theorem (MVT).

(b) Let $f : [0, 1] \to \mathbb{R}$ differentiable.
Assume that for all $x \in (a, b)$ we have that $f'(x) = 0$. Let $0 < c_1 < c_2 < 1$. Use MVT to show that $f(c_1) = f(c_2)$ and conclude that $f(x)$ is constant on $[a, b]$.

3. Calculate, if the limit exists

$$\lim_{(x,y)\to(0,0)} \frac{x \tan y}{x^2 + y^2}.$$

4. Let S_n be given by

$$S_n = \sum_{k=1}^{n} \frac{2k-1}{n^2}\frac{n}{k}.$$

We want to interpret S_n as a Riemann sum of a function $f(x)$ on the interval $(0, 1]$ with $\Delta x_k = \dfrac{2k-1}{n^2}$.

(a) For $n = 6$ depict the corresponding partition of $[0, 1]$ and determine $x_0, x_1, x_2, x_3, x_4, x_5, x_6$.

(b) For general n determine the corresponding partition $P_n = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ of the interval $[0, 1]$.

(c) Determine $f : (0, 1] \to \mathbb{R}$ such that S_n is a Riemann sum for $f(x)$ on $(0, 1]$.

(d) Now calculate

$$\lim_{n\to\infty} S_n.$$

5. Calculate the average value of $f(x) = x^8 \ln(x^3)$ on the interval $[1, \sqrt[3]{2}]$.

6. Let $f(x) = xe^{x^2}$.

   (a) Give the Taylor Series expansion of $f(x)$ about 0 up to and including the term of degree five.

   (b) Determine the radius of convergence of the Taylor series of $f(x)$ about 0.

Points: **Ex 1**, a: 3, b: 4, c: 3, d: 5, **Ex 2**: a: 4, b: 5, **Ex 3**: 8, **Ex 4**: a: 3, 4: 4, d:4. **Ex 5**: 8, **Ex 6**: a: 5, b: 4.