Mathematics B2: Newton

Solution to test of November 29, 2013

1. (a) Split limit in left and right [$\frac{1}{2}$ pt]

   Now $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + 2\sqrt{x} = 0$ [$\frac{1}{2}$ pt]

   and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{x}{x^2+1} = 0$ [$\frac{1}{2}$ pt]

   Hence $\lim_{x\to 0} f(x) = 0$ [$\frac{1}{2}$ pt]

   Since $f(0) = 0$, f is continuous at 0. [1 pt]

   (b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x + 2\sqrt{x}) = \infty$ [$\frac{1}{2}$ pt]

   $\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \dfrac{x}{x^2+1} = \lim_{x\to -\infty} \dfrac{\frac{1}{x}}{1+\frac{1}{x^2}}$ [1 pt]

   $= 0$ (correct answer, without motivation) [$\frac{1}{2}$ pt]

   (c) Formulate the procedure for finding extrema explicitly, 1 point divided as:

   From (a) we know that f is continuous on $[-2, 2]$ [$\frac{1}{2}$ pt]

   The candidates for absolute extrema are the points x where:

   * $f'(x)$ does not exist
   * $f'(x) = 0$
   * $x = -2$ and $x = 2$

   [$\frac{1}{2}$ pt]

   Calculations: f' does not exist at 0 (no proof required) [$\frac{1}{2}$ pt]

   $f'(x) = 0$, split in:

   $x > 0 : f(x) = x + 2\sqrt{x}$, so $f'(x) = 1 + \frac{1}{\sqrt{x}} \neq 0$ [$\frac{1}{2}$ pt]

   $x < 0 : f(x) = \frac{x}{x^2+1}$, so $f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2}$ [$\frac{1}{2}$ pt]

   Hence $f'(x) = 0 \leftrightarrow x^2 = 1 \leftrightarrow x = -1 (x < 0)$ [$\frac{1}{2}$ pt]

   Now $f(-2) = -\frac{2}{5}, f(-1) = -\frac{1}{2}, f(0) = 0, f(2) = 2 + 2\sqrt{2}$, [$\frac{1}{2}$ pt]

   so $-\frac{1}{2}$ is the absolute minimum, $2 + 2\sqrt{2}$ the absolute maximum. [$\frac{1}{2}$ pt]

2. $\lim_{x\to 0}(1+\frac{x}{2})^{\frac{1}{x}} = \lim_{x\to 0}\exp(\frac{\ln(1+\frac{x}{2})}{x})$ [1 pt]

First calculate $\lim_{x\to 0}\frac{\ln(1+\frac{x}{2})}{x}$

Type $\frac{0}{0}$, hence L'Hôpital can be applied [$\frac{1}{2}$ pt]

Hence $\lim_{x\to 0}\frac{\ln(1+\frac{x}{2})}{x} = \lim_{x\to 0}\frac{\frac{\frac{1}{2}}{1+\frac{x}{2}}}{1} = \frac{1}{2}$ [1 pt]

It follows that $\lim_{x\to 0}(1+\frac{x}{2})^{\frac{1}{x}} = e^{\frac{1}{2}}$ [$\frac{1}{2}$ pt]

3. (a) Use polar coordinates: $x = r\cos\theta, y = r\sin\theta$ [$\frac{1}{2}$ pt]

   leads to $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+}\frac{r^2\cos^2(\theta)r\sin(\theta)}{r^2}$ [$\frac{1}{2}$ pt]

   $= \lim_{r\to 0^+} r\cos^2(\theta)\sin(\theta) = 0,$ [$\frac{1}{2}$ pt]

   some motivation that the result is independent of θ [$\frac{1}{2}$ pt]

   Since $f(0,0) = 0$, f is continuous at $(0,0)$. [1 pt]

   (b) Equation of the tangent plane (either implicit in solution, or explicit):

   $z - z_0 = f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$ (*) [1 pt]

   Here: $x_0 = 1, y_0 = 1, z_0 = \frac{1}{2}$ [$\frac{1}{2}$ pt]

   and:

   $f_x(x,y) = \frac{2xy(x^2+y^2) - 2x(x^2y)}{(x^2+y^2)^2}$ [$\frac{1}{2}$ pt]

   $f_y(x,y) = \frac{x^2(x^2+y^2) - 2y(x^2y)}{(x^2+y^2)^2}$ [$\frac{1}{2}$ pt]

   Hence (*) becomes:

   $z - \frac{1}{2} = \frac{1}{2}(x-1)$ (or $z = \frac{1}{2}x$) [$\frac{1}{2}$ pt]