Exam Stochastic Differential Equations (Mastermath SDE)
30-05-2016; 10:00 – 13:00.

– Phones / advanced calculators are not allowed

– Please answer exercises 1,3 and 2,4,5 on separate pages.

1. **Exercise**

(2) a. Prove that if M is a continuous local martingale such that $\mathbb{E}(\sup_{s\geq 0}|M_s|) < \infty$, then M is a martingale.

(3) b. Define uniform integrability of a collection of measurable functions.
Prove that for an integrable random variable X defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the collection

$$\{\mathbb{E}(X|\mathcal{G}) : \mathcal{G} \text{ sub-sigma algebra of } \mathcal{F}\}$$

is uniformly integrable.

(3) c. Let M be a L^2 bounded continuous martingale with $M_0 = 0$ and $H \in L^2(K \bullet M)$ resp. $H \cdot K, K \in L^2(M)$.

1. Show associativity, i.e. $H \bullet (K \bullet M) = (H \cdot K) \bullet M$.
2. Demonstrate that $\left\langle \int_0^{\cdot} H_s dM_s \right\rangle_t = \int_0^t H_s^2 d\langle M \rangle_s$.

(5) 2. **Exercise**

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two independent $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions with $X_0 = Y_0 = 0$, $(H_t)_{t\geq 0}$ a progressively measurable process and

$$B_t = \int_0^t \cos(H_s)\, dX_s + \int_0^t \sin(H_s)\, dY_s$$

$$\hat{B}_t = \int_0^t \sin(H_s)\, dX_s - \int_0^t \cos(H_s)\, dY_s.$$

Show that B and $\hat{B}$ are two independent Brownian motions w.r.t $(\mathcal{F}_t)_{t\geq 0}$.

See next page.

3. **Exercise**

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $X \equiv (X_t)_{t\geq 0}$ a continuous adapted process satisfying

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dB_s, \qquad t \geq 0,$$

where $x_0 \in \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ is continuous. Suppose $a < x_0 < b$. In the following you will derive the probability distribution of X_T where T is the exit time, for X, of the interval (a, b), i.e., $T = \inf\{t \geq 0 : X_t \notin (a, b)\}$.

(1) a. Define the level-u crossing time of X as $T_u = \inf\{t \geq 0 : X_t = u\}$. Explain why $T = T_a \wedge T_b$.

(2) b. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Argue that $f(X_t)$ and $f(X_{t\wedge T})$ are semimartingales by deriving their explicit decompositions.

(2) c. Let M be a local martingale defined by

$$M_t = \int_0^{t\wedge T} f'(X_s)\, \sigma(X_s)\, dB_s, \quad t \geq 0.$$

Show that $\mathbb{E}(\langle M \rangle_t) < \infty$ for all $t > 0$ and M is a L^2-martingale.

(1) d. Show using (b) and (c) that

$$\mathbb{E}\left(f(X_{t\wedge T})\right) = f(x_0) + \frac{1}{2}\,\mathbb{E}\left(\int_0^{t\wedge T} f''(X_s)\, \sigma^2(X_s)\, ds\right).$$

Let g and h be functions defined on $[a, b]$ by

$$g(y) = \int_a^y \frac{1}{\sigma^2(z)}\, dz \quad \text{and} \quad h(x) = \int_a^x g(y)\, dy - \frac{x-a}{b-a} \int_a^b g(y)\, dy.$$

(1) e. Show that $h''(x) = \frac{1}{\sigma^2(x)}$ for $x \in (a, b)$ and $h(x) = 0$ for $x \in \{a, b\}$.

(2) f. Show using the previous parts that for all $t > 0$

$$\mathbb{E}(t \wedge T) = 2\mathbb{E}(h(X_{t\wedge T})) - 2h(x_0).$$

(1) g. Conclude that for $t > 0$,

$$\mathbb{E}(t \wedge T) \leq 4 \sup_{x\in[a,b]} |h(x)| < \infty$$

and $\mathbb{E}(T) < \infty$.

(2) h. Use (g) to conclude

$$\mathbb{E}(T) = -2\int_a^{x_0} g(y)\, dy + 2\frac{x_0 - a}{b-a} \int_a^b g(y)\, dy.$$

(3) i. Call now $w(x) = h(x) + \frac{b-x}{b-a}$ for $x \in [a, b]$. Use (d) to show

$$\mathbb{P}(X_T = a) = \frac{b - x_0}{b-a} \quad \text{and} \quad \mathbb{P}(X_T = b) = \frac{x_0 - a}{b-a}.$$

See next page.

4. **Exercise**

Let (B_t) be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

(4) a. Consider the stochastic differential equation (SDE)

$$dX_t = -\alpha\, X_t\, dt + \sigma\, dB_t, \quad t > 0,$$

with the initial condition $X_0 = x_0 \in \mathbb{R}$ and where α, σ positive constants. Derive the solution to the SDE by considering a solution of the form

$$X_t = a(t) \left\{ x_0 + \int_0^t b(s)\, dB_s \right\}.$$

(2) b. Determine the SDE satisfied by the process

$$Y_t = \exp\left\{ X_t - \frac{\eta}{\alpha} \right\}$$

where $\eta \in \mathbb{R}$.

(2) c. Use (a) and (b) to find the explicit form of a geometric mean reverting process satisfying the SDE

$$dr_t = r_t\, (\theta - \alpha\, \ln r_t)\, dt + \sigma\, r_t\, dB_t, \quad t > 0$$

where θ, α, σ are positive constants and $r_0 = 1$.

(4) 5. **Exercise**

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, $\{B_t, t \geq 0\}$ a standard $(\mathcal{F}_t)$-Brownian motion. Suppose (ν_t), (μ_t) and (σ_t) are continuous adapted processes.
Consider the stochastic process X_t satisfying

$$X_t = x + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dB_s, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}.$$

Construct, under appropriate conditions on (ν_t), (μ_t) and (σ_t), a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), \mathbb{P})$ such that X has the following representation under $\mathbb{Q}$:

$$X_t = x + \int_0^t \nu_s\, ds + \int_0^t \sigma_s\, d\tilde{B}_s, \quad 0 \leq t \leq T,$$

where $(\tilde{B}_t)$ is a $\mathbb{Q}$-BM.

[Hint: Express X_t (under $\mathbb{P}$) in the desired form for some $\tilde{B}_t$.]

Number of points can be found next to the questions; the grade will be calculated as follows:

$$\text{Grade} = \frac{\text{Number of points}}{40} \times 9 + 1.$$

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