Exam Stochastic Differential Equations (3TU)

June 15, 2009

The exam is closed book and consists of five problems with altogether fourteen items. The items are graded with 1, 2 or 4 points yielding a total of 25 points.

1. An urn contains b black and r red balls. A ball is drawn at random. It is replaced and, moreover, one ball of the same color is added. A new random drawing is made from the urn (now containing $r + b + 1$ balls), and this procedure is repeated. For $n = 1, 2, \ldots$, define the random variables X_n as follows: $X_n = 1$ if the nth drawing results in a red ball and $X_n = 0$ otherwise. Let Z_n be the fraction of red balls in the urn after the nth drawing, $n = 1, 2, \ldots$ and $Z_0 = r/(r + b)$.

   (a) **(1 pt)** Show that

   $$Z_n = \frac{r + \sum_{i=1}^n X_i}{r + b + n}.$$

   (b) **(2 pt)** Show that the sequence $\{Z_n : n \geq 0\}$ is a martingale with respect to the sequence $\{X_n : n \geq 1\}$.

   (c) **(1 pt)** Explain carefully according to which Theorem the sequence $\{Z_n : n \geq 0\}$ converges almost surely to a limit Z_∞.

   (d) **(1 pt)** Prove that

   $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n E[X_i] = E[Z_\infty],$$

   with Z_∞ defined as above.

2. Let B_t be a standard Brownian motion with filtration $(\mathcal{F})_t$. Define for $A, B > 0$,

   $$\tau = \min\{t : B_t = -B \text{ or } B_t = A\}.$$

   You may assume that τ has finite moments of all orders.

(a) **(2 pt)** Show that $B_{t\wedge\tau}$ is a martingale (you may refer to a Theorem). Show that the random variables $|B_{t\wedge\tau}|$, $t \geq 0$, are uniformly bounded by a constant.

(b) **(2 pt)** Show how $E[B_\tau]$ can be calculated from (a) and (b). Give also the probability distribution of B_τ.

Define

$$M_t = B_t^2 - t.$$

(d) **(1 pt)** Show that M_t is a martingale.

(e) **(1 pt)** Use the martingale M_t to prove that $E[\tau] = AB$.

3. Let B_t be a standard Brownian motion. Define the Gaussian processes

$$X_t = \int_0^t u \, \mathrm{d}B_u \text{ and } Y_t = \int_0^t B_u \, \mathrm{d}u, \quad t \in [0, T].$$

   (a) **(2 pt)** Calculate the stochastic differential $\mathrm{d}X_t Y_t$.

   (b) **(2 pt)** Calculate the covariance of X_t and Y_t.

4. (a) **(2 pt)** Solve the stochastic differential equation

$$\begin{aligned} \mathrm{d}Y_t &= (\theta - aY_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t \\ Y_0 &= y_0 \end{aligned}$$

   where a, σ are positive parameters and $\theta \in \mathbb{R}$.
   Hint: Let $Z_t = Y_t - \frac{\theta}{a}$, $t \geqslant 0$.

   (b) **(2 pt)** Let $X_t = e^{Y_t}, t \geqslant 0$, where Y_t is given in (a). Determine the stochastic differential equation satisfied by $\{X_t, t \geqslant 0\}$.

   (c) **(2 pt)** Let $\{r_t, t \geqslant 0\}$ satisfy

$$dr_t = r_t(\eta - a \log r_t)\,\mathrm{d}t + \sigma r_t\,\mathrm{d}B_t$$

   where η, a, σ are positive parameters. Solve this equation using (a) and (b).

5. **(4 pt)** Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $\{B_t, t \geqslant 0\}$ a standard Brownian motion with $\mathcal{F}_t = \sigma\{B_s; 0 \leqslant s \leqslant t\}$. Suppose that X_t satisfies the stochastic differential equation

$$\begin{aligned} dX_t &= \mu X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t, \quad 0 \leqslant t \leqslant T \\ X_0 &= x_0 \end{aligned}$$

and Y_t evolves deterministically as

$$\begin{aligned}\dot{Y}_t &= rY_t \\ Y_0 &= y_0\end{aligned}$$

Using Girsanov theorem, construct a probability measure under which $\tilde{X}_t \equiv \frac{X_t}{Y_t}$, $0 \leqslant t \leqslant T$ is an $\mathcal{F}_t$ martingale.