EXAM STOCHASTIC DIFFERENTIAL EQUATIONS (3TU)
June 11, 2012

Grading: [1+2+3+1] + [1+2+1+1] + [1+1+2+1] + [2+(2+1)+(1+1)] + [2]

1. Let S_n be simple symmetric random walk with $S_0 = 0$. Let

$$\tau = \min\{n \geq 1 : S_{n+1} = S_n + 1\} \qquad \text{and} \qquad \rho = \tau + 1.$$

   (a) Is τ a stopping time? Is ρ a stopping time?
   (b) Calculate $\mathbb{E}[\rho]$.
   (c) Use the Stopping Time Theorem to show that $\mathbb{E}[S_\rho] = 0$.
   (d) Calculate $\mathbb{E}[S_\tau]$.

   [You may use without further derivation that: $\sum_{n=1}^{\infty} nr^n = r(1-r)^{-2}$ for $r \in (-1, 1)$.]

2. An urn contains b black and r red balls. A ball is drawn at random. It is replaced and, moreover, one ball of the same color is added. A new random drawing is made from the urn (now containing $r + b + 1$ balls), and this procedure is repeated. For $n = 1, 2, \ldots$, define the random variables X_n as follows: $X_n = 1$ if the nth drawing results in a red ball and $X_n = 0$ otherwise. Let Z_n be the fraction of red balls in the urn after the nth drawing, $n = 1, 2, \ldots$ and $Z_0 = r/(r + b)$.

   (a) Show that

$$Z_n = \frac{r + \sum_{i=1}^{n} X_i}{r + b + n}.$$

   (b) Show that the sequence $\{Z_n : n \geq 0\}$ is a martingale with respect to the sequence $\{X_n : n \geq 1\}$.
   (c) Explain carefully according to which theorem the sequence $\{Z_n : n \geq 0\}$ converges almost surely to a limit Z_∞.
   (d) Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} E[X_i] = E[Z_\infty],$$

   with Z_∞ defined as above.

3. Let $\{B_t : t \geq 0\}$ be standard Brownian motion, $\lambda \geq 0$ and

$$Z_t = \exp(\sqrt{2\lambda} B_t - \lambda t), \quad t \geq 0.$$

   Define for $a > 0$

$$\tau = \inf\{t : B_t = a\}.$$

   You may assume that $\mathbb{P}(\tau < \infty) = 1$.

   (a) Show (from the first principle) that Z_t is a martingale with respect to the filtration $\mathcal{F}_t$ of the Brownian motion B_t.

(b) Show that the sequence $Z_{\tau\wedge n}$ is uniformly bounded by a constant.

(c) Conclude from (a) and (b) that

$$\mathbb{E}\left[e^{-\lambda\tau}\right] = e^{-a\sqrt{2\lambda}}.$$

(d) Conclude from (c) that

$$\mathbb{E}\left[\tau^{-1}\right] = a^{-2}.$$

Hint: Use the identity $x^{-1} = \int_0^\infty e^{-\lambda x}\,\mathrm{d}\lambda$ for $x > 0$ and recall that the expectation of an exponential r.v. with density $ae^{-ax}\mathbf{1}_{[0,\infty)}(x)$ $(a > 0)$ is equal to a^{-1}.

4. Let (B_t) be a standard Brownian motion and

$$dX_t = X_t dB_t, \quad X_0 = 1. \tag{1}$$

Define

$$Z_t = X_t e^{-\int_0^t B_s^2 ds}, \quad 0 \le t \le 1. \tag{2}$$

(a) Apply Itô's formula to show that

$$dZ_t = Z_t(dB_t - B_t^2 dt), \quad Z_0 = 1. \tag{3}$$

(b) Find the solution X_t satisfying the SDE (1) and use it to show that $E(Z_t^2) \le e^t$.

[You may just propose a solution to the SDE and appeal to the uniqueness theorem.]

(c) If one wants to consider the "integrated version" of the SDE (3) on its own, one needs to make sure that both of the following hold.

$$(i)\ E\left[\left(\int_0^1 Z_t dB_t\right)^2\right] < \infty \quad \text{and} \quad (ii)\ E\left[\int_0^1 Z_t B_t^2 dt\right] < \infty,$$

Use (b) to verify that indeed (i) and (ii) hold if Z_t is as given in (2).

[You may use the fact that if $Y \sim N(0,\sigma^2)$, then $E(Y^4) = 3\sigma^4$.]

5. Let

$$X_t = e^{-\frac{1}{2}t}e^{B_t} \tag{4}$$

under a measure $\mathbb{P}$ on $C[0,T]$ where B_t is a $\mathbb{P}$-Brownian motion. Let $\mathbb{Q}$ be a measure (on $C[0,T]$) defined by :

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}\left[\mathbb{I}_A X_T\right].$$

Show that under measure $\mathbb{Q}$,

$$X_t = e^{\frac{1}{2}t}e^{\tilde{B}_t} \tag{5}$$

where $\tilde{B}_t$ is a $\mathbb{Q}$-Brownian motion.

Hint: Write/express X_t as given in (4) in the form of (5). Apply Girsanov theorem to show that everything falls into places.

Solution

1. (a) $\{\tau = 1\} = \{S_1 = 1, S_2 = 2\} \cup \{S_1 = -1, S_2 = 0\}$, so $\{\tau = 1\} \notin \sigma(S_1)$. Hence τ is not a stopping time. On the other hand,

$$\begin{aligned}\{\rho = n+1\} \;=\; \{\tau = n\} \;=\; & \{S_1 = 1, \ldots, S_n = -n+2, S_{n+1} = -n+3\} \\ & \cup \{S_1 = -1, \ldots, S_n = -n, S_{n+1} = -n+1\}.\end{aligned}$$

So $\{\rho = n+1\} \in \sigma(S_1, \ldots, S_{n+1})$ and hence ρ is a stopping time.

(b)

$$\begin{aligned}\mathbb{P}(\rho = n+1) &= \mathbb{P}(S_1 = 1, \ldots, S_n = -n+2, S_{n+1} = -n+3) \\ &\quad + \mathbb{P}(S_1 = -1, \ldots, S_n = -n, S_{n+1} = -n+1) \\ &= (1/2)^{n+1} + (1/2)^{n+1} \;=\; (1/2)^n.\end{aligned}$$

So

$$\mathbb{E}(\rho) = \sum_{n=1}^{\infty} (n+1) \left(\frac{1}{2}\right)^n = 3.$$

(c) By 1(b), ρ is finite almost surely, so S_ρ is well-defined and $S_\rho = \lim_{n\to\infty} S_{\rho\wedge n}$. By the Stopping Time Theorem we have $\mathbb{E}[S_{\rho\wedge n}] = \mathbb{E}[S_0] = 0$. Since $|S_{\rho\wedge n}| \le \rho$, it follows from 1(b) and the dominated convergence theorem that

$$\mathbb{E}[S_\rho] = \mathbb{E}[\lim_{n\to\infty} S_{\rho\wedge n}] = \lim_{n\to\infty} \mathbb{E}[S_{\rho\wedge n}] = 0.$$

This can also be seen directly by noting that

$$\begin{aligned}\mathbb{E}[S_\rho] &= \sum_{n=1}^{\infty} \mathbb{E}\left[S_\rho \mathbf{1}_{\{\rho = n+1\}}\right] \\ &= \sum_{n=1}^{\infty} \{(-n+3)\mathbb{P}(S_1 = 1, \ldots, S_n = -n+2, S_{n+1} = -n+3) \\ &\qquad + (-n+1)\mathbb{P}(S_1 = -1, \ldots, S_n = -n, S_{n+1} = -n+1)\} \\ &= \sum_{n=1}^{\infty} (4-2n) \left(\frac{1}{2}\right)^{n+1} = 0.\end{aligned}$$

(d) Note that $S_\rho = S_{\tau+1} = S_\tau + 1$. So $\mathbb{E}[S_\tau] = -1$.

2. (a) Note that the total number of red balls after the nth drawing is $r + \sum_{i=1}^n X_i$. The total number of balls in the urn after n drawings is $b + r + n$. So the fraction of red balls in the urn after the nth drawing is given by $Z_n = (r + \sum_{i=1}^n X_i)/(b + r + n)$.

(b) From (a) we have that $Z_n = f_n(X_1, \ldots, X_n)$ where $f_n(x_1, \ldots, x_n) = \frac{(r+n+\sum_{i=1}^n x_i)}{(b+r+n)}$. The expectation of Z_n is finite since $|Z_n| \le 1$.

$$\mathbb{E}(Z_n | X_1, \ldots, X_{n-1}) = \frac{r + \sum_{i=1}^{n-1} X_i}{r+b+n} + \frac{1}{r+b+n}\mathbb{E}(X_n | X_1, \ldots, X_{n-1})$$

Now

$$\frac{r + \sum_{i=1}^{n-1} X_i}{r+b+n} = \left(1 - \frac{1}{r+b+n}\right) Z_{n-1}$$

and

$$\mathbb{E}(X_n|X_1,\ldots,X_{n-1}) = Z_n.$$

It follows that $\mathbb{E}(Z_n|X_1,\ldots,X_{n-1}) = Z_{n-1}$ and Z_n is a martingale.

(c) It follows from $|Z_n \leq 1$ that $\mathbb{E}(|Z_n|) \leq 1$. So the statement follows from the Bounded Martingale Convergence Theorem.

(d) The sequence Z_n is uniformly integrable since $|Z_n| \leq 1$, so Z_n converges in L^1 to Z_∞. In particular, $\lim \mathbb{E}[Z_n] = \mathbb{E}[Z_\infty]$. From 2(a) we have, $\sum_{i=1}^n X_i = (r+b+n)Z_n - r$. Hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i = \mathbb{E}[Z_\infty].$$

3. (a) Clearly Z_t is $\mathcal{F}_t$-measurable. Integrability of Z_t follows from the fact that for a standard normal r.v. U, $\mathbb{E}(\exp(xU)) = \exp(x^2/2)$.
Note further that for $s<t$ we have $Z_t = Z_s\, e^{\sqrt{2\lambda}(B_t-B_s)-\lambda(t-s)}$. So

$$\mathbb{E}(Z_t|\mathcal{F}_s) = Z_s\mathbb{E}\left[e^{\sqrt{2\lambda}(B_t-B_s)-\lambda(t-s)}\right] = Z_s\mathbb{E}\left[e^{\sqrt{2\lambda}\sqrt{t-s}\,U-\lambda(t-s)}\right]$$

where $U \sim N(0,1)$. Since $\mathbb{E}\left[e^{\sqrt{2\lambda}\sqrt{t-s}\,U-\lambda(t-s)}\right] = 1$, it follows that Z_t is a martingale.

(b) Note that for $n \leq \tau$, $Z_n \leq a$. Then for all n, $Z_{\tau\wedge n} \leq \exp(a\sqrt{2\lambda})$.

(c) τ is a stopping time. So by the stopping time theorem for continuous time parameter martingales we have that $\mathbb{E}[Z_{\tau\wedge n}] = \mathbb{E}[Z_0] = 1$. Since $\mathbb{P}(\tau<\infty)=1$, we have $\lim_{n\to\infty} Z_{\tau\wedge n} = \exp(a\sqrt{2\lambda}-\lambda\tau)$. By 3(b) and the dominated convergence theorem we get $\mathbb{E}[\exp(a\sqrt{2\lambda}-\lambda\tau)] = 1$. Multiplication with $\exp(-a\sqrt{2\lambda})$ gives the result.

(d) Since $\mathbb{P}(0<\tau<\infty)=1$, we get $\mathbb{E}[\tau^{-1}] = \mathbb{E}\left[\int_0^\infty e^{-\lambda\tau}\,\mathrm{d}\lambda\right]$ which equals by Fubini's theorem $\int_0^\infty \mathbb{E}\left[e^{-\lambda\tau}\right]\mathrm{d}\lambda = \int_0^\infty e^{-a\sqrt{2\lambda}}\,\mathrm{d}\lambda$. Substituting $x=\sqrt{2\lambda}$ we arrive at $\int_0^\infty xe^{-ax}\,\mathrm{d}x = a^{-1}\int_0^\infty xae^{-ax}\,\mathrm{d}x = a^{-2}$.

4. (a) Applying Itô's formula to $Z_t = f(t,X_t)$ where $f(t,x) = xe^{-g(t)}$ with $g(t) = \int_0^t B_s^2 ds$, we have

$$\begin{aligned} dZ_t &= f_t dt + f_x dX_t + \frac{1}{2} f_{xx} dX_t \cdot dX_t \\ &= X_t e^{-g(t)}\cdot(-g'(t))dt + e^{-g(t)}dX_t + 0 \\ &= X_t e^{-g(t)}(-B_t^2)dt + e^{-g(t)}X_t dB_t \\ &= Z_t(dB_t - B_t^2 dt) \end{aligned}$$

(b) An application of Itô formula shows that $X_t = e^{B_t-\frac{1}{2}t}$ is **a** solution to the SDE (1). The uniqueness theorem ensures that this is the (unique) solution.
Since $g(t) = \int_0^t B_s^2 ds \geq 0$ and as a result $Z_t = X_t e^{-g(t)} \leq X_t$, we have

$$E(Z_t^2) \leq E(X_t^2) = E[e^{2B_t - t}] = e^{-t}e^{\frac{1}{2}4t} = e^t.$$

(c) From Itô isometry it follows that

$$E\left[\left(\int_0^1 Z_t dB_t\right)^2\right] = \int_0^1 E(Z_t^2)dt \le \int_0^1 E(X_t^2)dt \le \int_0^1 e^t dt \le (e-1) < \infty.$$

Using Fubini and (two times) Cauchy-Schwartz we have

$$\begin{aligned} E\left[\int_0^1 Z_t B_t^2 dt\right] &= \int_0^1 E[Z_t B_t^2]dt \le \int_0^1 (E[Z_t^2])^{\frac{1}{2}} (E[B_t^4])^{\frac{1}{2}} dt \\ &\le \left(\int_0^1 E[Z_t^2]dt\right)^{\frac{1}{2}} \left(\int_0^1 E[B_t^4]dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 E[Z_t^2]dt\right)^{\frac{1}{2}} \left(\int_0^1 3t^2 dt\right)^{\frac{1}{2}} \\ &\le \sqrt{e-1} < \infty. \end{aligned}$$

5. Note that

$$X_t = e^{-\frac{1}{2}t} e^{B_t} = e^{\frac{1}{2}t} e^{B_t - t} = e^{\frac{1}{2}t} e^{\tilde{B}_t},$$

where $\tilde{B}_t = B_t - t = B_t - \int_0^t \mu\, ds$, with $\mu \equiv 1$. Let us use Girsanov theorem to find a new measure such that the new process $\tilde{B}_t$ becomes a BM. We obtain the new measure to be $\mathbb{Q}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^T \mu\, dB_s - \frac{1}{2}\int_0^T \mu^2\, ds} = e^{B_T - \frac{1}{2}T} = X_T.$$

This is thus the same measure given in the question, namely, $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}\left[\mathbb{I}_A X_T\right]$.

Hence we have proved that $X_t = e^{\frac{1}{2}t} e^{\tilde{B}_t}$, where $\tilde{B}_t$ is a $\mathbb{Q}$-BM.