EXAM STOCHASTIC DIFFERENTIAL EQUATIONS (3TU)
May 30, 2011

Grading: $2 + (1{+}1\frac{1}{2}) + 1\frac{1}{2} + 2 + 2$

1. Let $\gamma_1$ and $\gamma_2$ be independent standard normal random variables. Prove the identity
$$\mathbb{E}(1_{\{\gamma_1 \geq \gamma_2\}} \,|\, \gamma_1) = \Psi(\gamma_1),$$
where
$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^2)\, dt.$$

2. Let $0 < p < 1$ and suppose $(\eta_n)_{n\geq 1}$ is a sequence of independent, identically distributed random variables such that $\mathbb{P}(\eta_n = k) = (1-p)^{k-1}p$ for all $k = 1, 2, \dots$ For $n = 1, 2, \dots$ let $\mathcal{F}_n := \sigma(\eta_1, \dots, \eta_n)$ and
$$\xi_n := \exp(\eta_1 + \cdots + \eta_n - na).$$
   a. For which values of $p \in (0,1)$ and $a \in \mathbb{R}$ is the sequence $(\xi_n)_{n\geq 1}$ a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$?
   b. For these values of $p$ and $a$, show that the martingale $(\xi_n)_{n\geq 1}$ converges almost surely. Determine the almost sure limit (*Hint:* Consider the behaviour of $\eta_1 + \cdots + \eta_n$ as $n \to \infty$). Do we have convergence in $L^1$?

3. Let $(M_n)_{n\geq 0}$ be a nonnegative submartingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ and let $\tau$ be a stopping time with respect to the same filtration such that $\tau < \infty$ almost surely. Prove that
$$\mathbb{E}M_\tau \leq \underline{\lim}_{n\to\infty} \mathbb{E}M_n.$$
Interpret this in terms of your winnings in a gambling game.

4. For $0 < T < \infty$, define
$$G := \int_0^T B_t^2\, dt.$$
Identify the process $g \in \mathcal{H}^2[0,T]$ such that
$$G = E[G] + \int_0^T g(t)\, dB_t \quad \text{a.s.}$$
*Hint:* Apply Itô's formula to $Y_t := tB_t^2$.

5. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and set $\mathcal{F}_t := \sigma(B_s\,;\, 0 \leq s \leq t)$, $t \geq 0$. Suppose $(X_t)_{0\leq t\leq T}$ satisfies the stochastic differential equation
$$\begin{aligned} dX_t &= \mu X_t\, dt + \sigma X_t\, dB_t, \quad 0 \leq t \leq T,\\ X_0 &= x_0, \end{aligned}$$
and $(Y_t)_{0\leq t\leq T}$ evolves deterministically as
$$\begin{aligned} \dot{Y}_t &= rY_t, \quad 0 \leq t \leq T,\\ Y_0 &= y_0. \end{aligned}$$
where $\mu, \sigma, r, x_0$ and $y_0$ are positive constants, and $\mu$ is greater than $r$. Using the Girsanov theorem, construct a probability measure under which $\tilde{X}_t \equiv \dfrac{X_t}{Y_t}$, $0 \leq t \leq T$, is an $\mathcal{F}_t$-martingale.