EXAM STOCHASTIC DIFFERENTIAL EQUATIONS (Mastermath)
June 3rd, 2013

1.

(a) State and prove the tower property of the conditional expectation. (6 p)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be convex and increasing. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. Let $(M_n)_{n\geq 1}$ be a submartingale such that for each $n \geq 1$, $X_n := f(M_n) \in L^1(\Omega)$. Show that $(X_n)_{n\geq 1}$ is a submartingale as well. (6 p)

answer

1a: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X \in L^1$ and let $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ be σ-algebra's. Then $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$. Proof: Note that $\mathbb{E}(X|\mathcal{H})$ is H-measurable. Moreover, for all $H \in \mathcal{H}$ one has

$$\int_{\mathcal{H}} \mathbb{E}(X|\mathcal{H})\, d\mathbb{P} = \int_{\mathcal{H}} X\, d\mathbb{P} = \int_{\mathcal{H}} \mathbb{E}(X|\mathcal{G})\, d\mathbb{P}.$$

Now the result follows from the definition of the conditional expectation.

1b: $\mathbb{E}(M_n|\mathcal{F}_{n-1}) \geq M_{n-1}$. Therefore, using conditional Jensen's inequality and the fact that f is increasing we find

$$\mathbb{E}(X_n|_{\mathcal{F}_{n-1}}) \geq f(\mathbb{E}(M_n|\mathcal{F}_{n-1})) \geq f(M_{n-1}) = X_{n-1}.$$

2. Assume $(X_n)_{n\geq 1}$ is a sequence of independent random variables such that

$$\mathbb{E}(X_n) = \mathbb{E}(X_n^3) = 0, \quad \mathbb{E}(X_n^2) = 1 \quad \mathbb{E}(X_n^4) = \alpha.$$

Let $S_n = \displaystyle\sum_{j=1}^{n} X_j$ and $S_0 = 0$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and define

$$M_n = S_n^4 - 6nS_n^2 + (3-\alpha)n + 3n^2, \quad n \geq 0$$

(a) Show that $(M_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. (7 p)
Hint: Write $S_n = S_{n-1} + X_n$ and use the identity

$$(s+x)^4 = s^4 + 4s^3x + 6s^2x^2 + 4sx^3 + x^4.$$

Next assume $\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = 1/2$ and note that $\alpha = 1$. Let $A \in \mathbb{N} \setminus \{0\}$ and let $\tau = \inf\{n \geq 0 : |S_n| = A\}$. It is known that $\mathbb{E}(\tau) = A^2$ and that τ has finite moments of all orders and you may use both these facts below.

(b) Show that $\mathbb{E}(M_\tau) = 0$. (7 p)
Hint: Use the stopping time theorem and dominated convergence.

(c) Derive that $\mathbb{E}(\tau^2) = \frac{5A^4 - 2A^2}{3}$. (5 p)

answer 2a Taking out what is known and using independence yields

$$
\begin{aligned}
\mathbb{E}(M_n|F_{n-1}) &= \mathbb{E}(S_n^4 - 6nS_n^2 + (3-\alpha)n + 3n^2|\mathcal{F}_{n-1}) \\
&= \mathbb{E}(S_{n-1}^4 + 4S_{n-1}^3X_n + 6S_{n-1}^2X_n^2 + 4S_{n-1}X_n^3 + X_n^4|\mathcal{F}_{n-1}) \\
&\quad - 6n\Big[\mathbb{E}(S_{n-1}^2 + 2S_{n-1}X_n + X_n^2|\mathcal{F}_{n-1})\Big] + (3-\alpha)n + 3n^2 \\
&= S_{n-1}^4 + 4S_{n-1}^3\mathbb{E}(X_n|\mathcal{F}_{n-1}) + 6S_{n-1}^2\mathbb{E}(X_n^2|\mathcal{F}_{n-1}) + 4S_{n-1}\mathbb{E}(X_n^3|\mathcal{F}_{n-1}) + \mathbb{E}(X_n^4|\mathcal{F}_{n-1}) \\
&\quad - 6n\Big[S_{n-1}^2 + 2S_{n-1}\mathbb{E}(X_n|_{\mathcal{F}_{n-1}}) + \mathbb{E}(X_n^2|\mathcal{F}_{n-1})\Big] + (3-\alpha)n + 3n^2 \\
&= S_{n-1}^4 + 6S_{n-1}^2 + \alpha - 6nS_{n-1}^2 - 6n + (3-\alpha)n + 3n^2 = M_{n-1}
\end{aligned}
$$

2b: First note that τ is a stopping time, because $\{\tau > n\} = \{|S_j| < A, 1 \le j \le n\}$. By the stopping theorem it follows that $(M_{n\wedge\tau})_{n\ge 0}$ is a martingale. Therefore, $\mathbb{E}(M_{n\wedge\tau}) = \mathbb{E}(M_0) = 0$. Given is that $\tau < \infty$ a.s. Therefore, $|S_{n\wedge\tau}| \le A$. It follows that

$$
|M_{n\wedge\tau}| \le A^4 - 6\tau A^2 + |3-\alpha|\tau + 3\tau^2, \quad n \ge 0
$$

Since the right-hand side is integrable and does not depend on $n \ge 0$, the dominated convergence theorem and $\lim_{n\to\infty} M_{n\wedge\tau} = M_\tau$ imply that

$$
\mathbb{E}(M_\tau) = \lim_{n\to\infty} E(M_{n\wedge\tau}) = 0.
$$

2c: Since $\tau < \infty$ a.s., one has $|S_\tau| = A$ almost surely and by 2b we have

$$
0 = \mathbb{E}(M_\tau = \mathbb{E}(A^4 - 6\tau A^2 + (3-\alpha)\tau + 3\tau^2).
$$

Therefore, using $\mathbb{E}(\tau) = A^2$ we find that $\mathbb{E}(\tau^2) = \frac{5A^4 - 2A^2}{3}$.

3. Assume $(Z_j)_{j\ge 1}$ are independent random variables with normal distribution and $\mathbb{E}(Z_j) = 0$ and $\mathbb{E}(Z_j^2) = 1$. Let $S_n = \sum_{j=1}^n Z_j$ and let $X_n = \exp(S_n - n^\alpha)$, where $\alpha > 0$ is a fixed parameter.

(a) Characterize those $\alpha > 0$ for which one has $\lim_{n\to\infty} X_n = 0$ in L^1. (6 p)

Hint: You may use the identity: $\mathbb{E}(e^{Z_n}) = e^{1/2}$.

(b) Characterize those $\alpha > 0$ for which one has $\lim_{n\to\infty} X_n = 0$ in probability. (6 p)

Answer

3a: By independence and $\mathbb{E}(e^{Z_j}) = e^{1/2}$ one has

$$
\mathbb{E}|X_n| = \mathbb{E}(X_n) = \mathbb{E}(\exp(S_n - n^\alpha)) = e^{-n^\alpha}\mathbb{E}\Big(\prod_{j=1}^n e^{Z_j}\Big) = e^{-n^\alpha}\prod_{j=1}^n \mathbb{E}(e^{Z_j}) = e^{-n^\alpha + \frac{n}{2}}.
$$

Therefore, $X_n \to 0$ in L^1 if and only if $-n^\alpha + \frac{n}{2} \to \infty$. This holds if and only if $\alpha \ge 1$.

3b. Recall that $X_n \to 0$ in probability if and only if for every $\varepsilon > 0$ one has $\lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) = 0$. It suffices to consider $\varepsilon \in (0, 1]$. For such ε we have

$$
\begin{aligned}
\mathbb{P}(|X_n| > \varepsilon) &= \mathbb{P}(S_n - n^\alpha > \log(\varepsilon)) \\
&= \mathbb{P}(S_n > \log(\varepsilon) + n^\alpha) \\
&= \mathbb{P}\Big(\frac{S_n}{n^{1/2}} > \frac{\log(\varepsilon) + n^\alpha}{n^{1/2}}\Big) \\
&= \Phi(p(\alpha, n)),
\end{aligned}
$$

where $p(\alpha, n) = \frac{\log(\varepsilon)+n^\alpha}{n^{1/2}}$ and $\Phi(x) = \mathbb{P}(Z_1 > x)$. Now $\lim_{n\to\infty} p(\alpha, n) = \infty$ if and only if $\alpha > 1/2$. If $\alpha \leq 1/2$, then $p(\alpha, n) \leq 1)$ and hence $\Phi(p(\alpha, n)) \geq \Phi(1)$. We can conclude that $X_n \to 0$ in probability if and only if $\alpha > 1/2$.

4. Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $a > 0$.
Define the processes X and Y by $X_t = a^{-1/2}B_{at}$ and $Y_t = B_{2t} - B_t$.

(a) Prove or disprove: $(X_t)_{t\geq 0}$ is a Brownian motion. (5 p)

(b) Prove or disprove: $(Y_t)_{t\geq 0}$ is a Brownian motion. (5 p)

(c) What are the mean and variance of $\int_0^T t^4 B_t \, dB_t$. Explain your answer. (7 p)

answer

4a. Clearly, (X_t) is a Gaussian process and it has continuous paths again. Moreover, $X_0 = 0$ and for all $t > 0$, $\mathbb{E}(X_t) = 0$ and for all $t \geq s \geq 0$,

$$\mathbb{E}(X_t X_s) = a^{-1}\mathbb{E}(B_{at}B_{as}) = a^{-1}\min\{at, as\} = s.$$

Therefore, a result from the book/lectures shows that $(X_t)_{t\geq 0}$ is a Brownian motion again.

4b. Note that

$$\mathbb{E}(Y_2 - Y_1)^2 = \mathbb{E}[B_4 - B_2 - (B_2 - B_1)]^2 = \mathbb{E}(B_4 - B_2)^2 - 2\mathbb{E}(B_4 - B_2)(B_2 - B_1) + \mathbb{E}(B_2 - B_1)^2 = 2 + 1 = 3.$$

This should be 1 for Brownian motion. Thus $(Y_t)_{t\geq 0}$ cannot be a Brownian motion.

4c By the Ito isometry one has

$$\mathbb{E}\Big|\int_0^T t^4 B_t \, dB_t\Big|^2 = \int_0^T \mathbb{E}t^8 B_t^2 \, dt = \int_0^T t^9 \, dt = \frac{1}{10}T^{10}.$$

It follows that $t^4 B_t$ defines a function in $\mathcal{H}^2$. Thus the Itô integral is a continuous time martingale starting at zero. Therefore, $\mathbb{E}\Big(\int_0^T t^4 B_t \, dB_t\Big) = 0$. Thus the mean is zero and variance $\frac{1}{10}T^{10}$.

5. Let (B_t) be a standard Brownian motion defined on the (filtered) probability space $(\Omega, \mathcal{F}\ (\mathcal{F}_t), P)$. For fixed parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ consider the SDE

$$dX_t = \mu dt + \sigma X_t dB_t, \qquad (*)$$

with the initial condition $X_0 = x_0 \in \mathbb{R}$.

(a) Consider the process $H_t = e^{-\sigma B_t + \frac{1}{2}\sigma^2 t}$. Show that H_t satisfies (7 p)

$$dH_t = -\sigma H_t dB_t + \sigma^2 H_t dt.$$

(b) Suppose X_t is a solution to the SDE $(*)$. Use the (Itô) product rule and (a) to show that (8 p)

$$d(H_t X_t) = \mu H_t dt.$$

(c) Use (b) and the definition of H to show that the solution of $(*)$ is given by (5 p)

$$X_t = x_0 e^{\sigma B_t - \frac{1}{2}\sigma^2 t} + \mu \int_0^t e^{\sigma(B_t - B_s) - \frac{1}{2}\sigma^2(t-s)} ds.$$

answer

5a: Let $f(x,t) = \exp\left(-\sigma x + \frac{1}{2}\sigma^2 t\right)$. Note that $f \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+)$ with

$$\frac{\partial f}{\partial x}(x,t) = -\sigma f(x,t), \qquad \frac{\partial^2 f}{\partial x^2}(x,t) = \sigma^2 f(x,t), \quad \text{and} \quad \frac{\partial f}{\partial t}(x,t) = \frac{1}{2}\sigma^2 f(x,t).$$

Applying Itô formula to $f(B_t,t) = H_t$ then leads to

$$\begin{aligned} dH_t &= df(B_t,t) = \frac{\partial f}{\partial t}(B_t,t)dt + f_x\frac{\partial f}{\partial x}(B_t,t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t,t)dt \\ &= \frac{1}{2}\sigma^2 f(B_t,t)dt - \sigma f(B_t,t)dB_t + \frac{1}{2}\sigma^2 f(B_t,t)dt \\ &= -\sigma H_t dB_t + \sigma^2 H_t dt, \end{aligned}$$

which was to be proven.

5b: Let X_t be the solution to $(*)$. Itô product rule (in box calculus notation) implies that

$$d(H_tX_t) = X_t dH_t + H_t dX_t + dH_t \cdot dX_t.$$

Using (a) and $(*)$ we then have

$$\begin{aligned} d(H_tX_t) &= X_t(-\sigma H_t dB_t + \sigma^2 H_t dt) + H_t(\mu dt + \sigma X_t dB_t) + dH_t \cdot dX_t \\ &= \sigma^2 X_t H_t dt + \mu H_t dt + (-\sigma H_t dB_t + \sigma^2 H_t dt)\cdot(\mu dt + \sigma X_t dB_t) \\ &= \sigma^2 X_t H_t dt + \mu H_t dt - \sigma^2 X_t H_t dt \\ &\qquad (\text{since } dB_t \cdot dB_t = dt \ \text{ and } \ dB_t \cdot dt = dt \cdot dB_t = dt \cdot dt = 0) \\ &= \mu H_t dt. \end{aligned}$$

5c: Using the integral form of $d(H_tX_t) = \mu H_t dt$ we have

$$H_tX_t = H_0X_0 + \int_0^t H_s X_s ds.$$

Using $H_0 = 1$ and $X_0 = x_0$ we get

$$X_t = x_0 H_t^{-1} + H_t^{-1}\int_0^t H_s X_s ds = x_0 H_t^{-1} + \int_0^t H_t^{-1} H_s X_s ds.$$

The result then follows from the definition of H_t.

6. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Set $\mathcal{F}_t := \sigma(B_s\,;\, 0 \leq s \leq t)$, $t \geq 0$. Suppose $(X_t)_{0\leq t\leq T}$ satisfies the stochastic differential equation

$$\begin{aligned} dX_t &= rX_t\,dt + \sigma X_t\,dB_t, \quad 0 \leq t \leq T, \\ X_0 &= x_0. \end{aligned}$$

where r, σ and x_0 are positive constants. Using the Girsanov theorem, construct a probability measure under which X_t is an $\mathcal{F}_t$-martingale. (10 p)

answer

For X_t to be a martingale we necessarily need to have the drift to be zero.

Note that $\theta_t = \frac{rX_t - 0}{\sigma X_t} = \frac{r}{\sigma}$, being a finite constant, is bounded. We can then use the Girsanov theorem to swap the drift by considering the new measure Q given by $\frac{dQ}{dP} = M_T$, where

$$M_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) = \exp\left(-\frac{r}{\sigma}B_t - \frac{1}{2}\frac{r^2}{\sigma^2}t\right).$$

Then under Q, $\tilde{B}_t = B_t + \frac{r}{\sigma}t$ is a BM and $dX_t = \sigma X_t d\tilde{B}_t$.

This implies that (under Q) $X_t = \exp\left(\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t\right)$, which is a Martingale.