Test T1 Differential Equations & Numerical Methods

Module	:	TW M6 Dynamical Systems (201500103)
Date	:	Friday December 22, 2017
Time	:	8:45 - 11:45 uur
Duration	:	180 min (In case of extra time: 225 min)
	:	30 min (In case only Numerical Methods is tested)
	:	150 min (In case only Differential Equations is tested)
Module-coördinator	:	H.G.E. Meijer
Teacher	:	H.G.E. Meijer
Test Type	:	Closed book
Supplements	:	None
Tools allowed	:	(Grafical) Calculator

Notices:

- Motivate your answers.
- This test consists of 3 pages, including this one, and contains 6 exercises.
- For this test you can get 36 points; i.e. grade = 1+points/4. The points for each exercises are mentioned below.
- If you only take Differential Equations, please skip Exc 6;
  If you only take Numerical Methods, hand in Exc 6 only. The grading is adjusted accordingly.
- Only use UT exam paper. Write your name and student number on each sheet of paper. Do not hand in your notes on scratch paper.

Subpoints:

1a	3	3a	2	4b	2	6a	2
1b	1	3b	3	4c	3	6b	2
2a	1	4c	3	5a	2	6c	2
2b1/2	4	4a	2	5b	4		

Grade = 1+points/4

Exercises Differential Equations

Exercise 1.

(a) Determine the solution of the first order ODE

$$\frac{dx}{dt} = \frac{1-5x(t)}{10+2t}, \quad \text{with} \quad x(0) = 1,$$

including the domain of existence.

(b) Determine $\lim_{t\to\infty} x(t)$

Exercise 2. We define the following matrix

$$A := \begin{pmatrix} -1 & -1 \\ 2 & -3 \end{pmatrix}.$$

(a) Compute the complex eigenvalues.

Now we want to compute e^{At}. Here you have two options (you get the points only once).

(b1) Use a standard approach.

(b2) For a 2x2 matrix A with complex eigenvalues $\lambda = \alpha \pm \beta i$ there is an explicit formula.

$$\Phi(t) = e^{-\alpha t}\left(\cos(\beta t)I + \frac{1}{\beta}(A-\alpha I)\sin(\beta t)\right)$$

If you check that Φ satifies the ODE $\Phi' = A\Phi$ with initial condition $\Phi(0) = I$ for general A, you may then use this formula to compute $e^{tA} = \Phi(t)$ for the A given. You will need the identity $A^2 = -2\alpha A - (\alpha^2+\beta^2)I$ for 2×2-matrices A.

Exercise 3. Consider the following system

$$\begin{cases} x' = & x - y, \\ y' = & x^2 - y. \end{cases}$$

(a) Determine the equilibria and classify their type.

(b) Determine a conserved quantity.

(c) Sketch the global phase portrait.
Hint: drawing nullclines first may just help you out.

T.O.P.

Exercise 4. Consider the following system

$$\begin{cases} x' = & x + 2y(1 - ay) - x(x^2 + y^2), \\ y' = & y + 2x(ay - 1) - y(x^2 + y^2), \end{cases}$$

with $a \in \mathbb{R}$ a parameter.

(a) Transform the system to polar coordinates.

(b) Determine a such that the system has three equilibria.

(c) Sketch the global phase portrait in the (x, y)-plane for $a = 0$, $a = 1$ and $a = 2$.

Exercise 5. Consider the following model for competing species

$$\begin{cases} x' = & x(3 - 2x - y), \\ y' = & y(2 - x - y). \end{cases}$$

(a) Show that $x, y \geq 0$ and $x + y \leq 3$ defines a trapping region.

(b) Determine the behaviour of a solution in the first quadrant with $x, y > 0$ as $t \to \infty$.

Exercises Numerical Mathematics

Exercise 6. Consider the boundary value problem:

$$y''(x) = \sqrt{x}\, y(x) + x, \tag{1}$$

with boundary conditions

$$y(1) = 1, \;\; y'(3) = 2. \tag{2}$$

We use a uniform grid on the interal $[1, 3]$ consisting of $n+1$ points, separated by a grid spacing h. The grid points are denoted $x_k, k = 0, \dots, n$, with $x_0 = 1, x_n = 3$. The corresponding $n+1$ values $\{y_k\}$ are approximations of $y(x_k)$.

(a) Show that the central difference

$$\Big(\delta_2 y\Big)_k = \frac{1}{h^2}\Big(y_{k+1} - 2y_k + y_{k-1}\Big)$$

approximates the second derivative $y''(x_k)$ in the location x_k with second order accuracy.

(b) Use the ghost cell method to create a second order accurate discretization of the problem in x_n - explicitly write down the discrete equation for y_n.

(c) Describe the total linear system of equations that needs to be solved to determine all values $\{y_k\}$.