MSc course 191551150 "Numerical techniques for PDEs" Final test, January 15, 2015, 08:45–10:45

The parts A and B may be graded separately. Therefore please write your answers for the A and B parts on separate sheets of paper. The use of calculators and other electronic devices is not allowed. Motivate all your answers.

Part A. Parabolic PDEs Consider a parabolic PDE

$$u_t - (D(u)u_x)_x = 0, \tag{1}$$

with initial and boundary conditions given, where $u = u(x,t)$ is unknown and $D(u) = u^\alpha$, $\alpha \geqslant 0$ is a constant parameter.

1pt **A1** Is (1) a nonlinear diffusion equation? Motivate your answer.

2pt **A2** Consider the case $\alpha = 0$ and formulate the θ-method for solving (1). What are the stability properties and the order of convergence of the θ-method for different values of θ? You do not have to prove or derive these properties.

3pt **A3** Assume $\alpha > 0$ and the PDE (1) is posed for $x \in [0,1]$. First, formulate a conservation property by integrating the equation over the space domain. After that, give a conservative central difference discretization for the term $(D(u)u_x)_x$ and explain why the conservation is preserved on the discrete level. Finally, based on this discretization, formulate an explicit in time numerical scheme for solving (1). Without carrying out a strict accuracy analysis, what could you say about the expected order of accuracy in this scheme?

Part B. Hyperbolic PDEs

1pt **B1** Consider the following hyperbolic PDE with some initial and boundary conditions given:

$$u_t + au_x = 0,$$

where $u = u(x,t)$ is unknown and $a = a(x,t)$ is given. Suppose a numerical scheme to solve the PDE satisfies the CFL condition. Based on this fact, can something be said on the stability of the scheme? Motivate your answer.

2pt **B2** Consider a scalar conservation law

$$u_t + (f(u))_x = 0,$$

where $u = u(x,t)$ is unknown and the following finite volume scheme for its solution:

$$(U_j^{n+1} - U_j^n)\Delta x + \frac{\Delta t}{2}\left[(f_{j+1/2}^n - f_{j-1/2}^n) + (f_{j+1/2}^{n+1} - f_{j-1/2}^{n+1})\right] = 0.$$

Explain how this finite volume scheme can be derived by double integrating the conservation law in space and in time over a finite volume cell. Without carrying out a strict accuracy analysis, what could you say about the expected order of accuracy in this scheme?

3pt **B3** Consider the second order hyperbolic PDE

$$u_{tt} - a^2 u_{xx} = 0,$$

with $u = u(x,t)$ unknown and $a = \text{const} > 0$ given, and the following numerical scheme for its solution:

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{(\Delta t)^2} - a^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2} = 0.$$

The CFL condition of the scheme is $|\nu| \leqslant 1$, where ν is defined as usual (see below). Carry out the first several steps of the Fourier analysis for this scheme namely:
(1) substitute the numerical Fourier mode $U_j^n = \lambda^n e^{ik\Delta x j}$ into the scheme and show that the amplification factor λ satisfies the relation

$$\lambda^2 + 2(2\nu^2 s^2 - 1)\lambda + 1 = 0, \qquad \nu = \frac{a\Delta t}{\Delta x}, \qquad s = \sin\frac{\xi}{2}, \qquad \xi = k\Delta x,$$

where k is the Fourier mode number.
(2) provide an expression for λ and a Fourier stability condition for the derived λ.

The grade for the test is determined as $G = 1 + 9P/12$ where P is the number of points earned.