Exam Stochastic Models (IEM: 202000425 / Math: 202001366)
Wednesday May 19, 13.45 - 16.45 hours.

Modelling and analysis of stochastic processes for IEM/Math (coord. Mes/Scheinhardt)

Teachers: Boucherie, Braaksma, Scheinhardt
Books, notes, etc. are **not** allowed. An ordinary (scientific) calculator is allowed, but a programmable or graphic calculator ('GR') is **not** allowed.

This exam consists of **four exercises**. Use **different sheets of paper** for exercises 1–2 and exercises 3–4, respectively. Put your name and student number on **each paper** you submit. **Motivate** all answers.

Total points: 90. Grade = 1 + obtained points / 10

1. (23 points) A repair center with a single repair-person has a fixed set of $N = 5$ customers, each with an installed product that is subject to breakdowns, after which it needs to be repaired. After repair an installed product is as good as a new product. Historical data reveals that the time between repair and breakdown of an installed product has an exponential distribution with mean $1/\nu$. Historical data also reveals that the amount of work required to repair an installed product has an exponential distribution with mean $1/\tau$. The repair center values both its repair-person and its clients. Therefore, it has decided that the repair-person may go off duty if the repair-person completes repairs and finds only one installed product that needs repairs. As the center also values its customers, it has decided that it will ask its repair-person to report on duty and start repairs if at least three installed products need repairs. Let $\{(X(t), Y(t)),\ t > 0\}$ be the stochastic process with states (i, j), where i is the number of installed products that need repairs, and j is the state of the repair-person (1 for on duty, 0 for off duty).

(a)–3 pt Why is $Y(t)$ required to make $\{(X(t), Y(t)),\ t > 0\}$ a Markov chain?

(b)–3 pt Assume that $X(t)$ is in state (3,1). What is the distribution of the time until $X(t)$ leaves state $(3, 1)$? What is the probability that in this state a breakdown occurs before a repair is completed? Motivate your answer.

(c)–2 pt Which states are in the same communicating class as state $(3, 1)$? Motivate your answer.

(d)–3 pt Give the transition diagram of $\{(X(t), Y(t)),\ t > 0\}$, including a motivation for the transition rates in this diagram.

(e)–6 pt Let $P(i, j)$ denote the steady state distribution of $\{(X(t), Y(t)),\ t > 0\}$. Give the equations to determine this steady state distribution, and solve these equations to determine $P(i, j)$ for the case where $1/\nu = 1$ hour and $1/\tau = 2$ hours.

P.T.O.

Express the answers to questions (f) and (g) below in terms of $P(i, j)$ (without using the numerical solution you found in (e) above).

(f)–2 pt What is the utilisation of the repair-person? Motivate your answer.

(g)–2 pt Give an expression for the average number of installed products waiting for repair. Motivate your answer.

(h)–2 pt What is the distribution of the consecutive time that all installed products are broken? Motivate your answer.

2. (22 points) Consider a network of 3 stations. Every station contains a single server, and all arriving customers arriving at a station may enter the station. Service is in order of arrival. The service times at stations 1 and 3 have exponential distributions with rates $\mu_1 = \mu$, and $\mu_3 = 2\mu$, respectively. The service times at station 2 are deterministic and have the value $1/(\alpha\mu)$ for some $\alpha > 0$. The arrivals to the network follow a Poisson process with rate λ. Each customer that arrives (from outside) independently selects either station 1 or station 2 with probability 1/2. A customer that completes service at station 1 is routed to station 3. Customers that complete service at stations 2 or station 3 leave the system.

(a)–3 pt Describe the arrival process to station 1. Motivate your answer.

(b)–3 pt What is the arrival rate to station 3? Motivate your answer.

(c)–3 pt Give the stability condition for the network. Motivate your answer.

(d)–8 pt Determine α such that in equilibrium the mean number of customers at stations 1 and 2 are equal. (Hint: use Pollackzek-Khintchine and Little)

(e)–5 pt Give the joint equilibrium distribution for the number of customers at stations 1 and 3. Motivate your answer.

Please use **different sheets of paper** for exercises 3–4 than for exercises 1–2.

3. (20 points) A famous but somewhat frivolous opera singer is scheduled to sing on 5 successive nights. If she is satisfied with her performance on a given night k (which happens with probability $p = 0.6$, independently of the previous history) she will sing on the following night (i.e., night $k+1$). If she is not satisfied, however, she declares that she will not sing on any of the future nights. In this case, the only way to possibly motivate her to perform on night $k+1$ is for the opera director to send her an expensive gift, costing €15,000. With probability $q = 0.45$ this gift will have the desired effect (i.e., the opera singer will perform on night $k+1$). If the gift does not have the desired effect, the missed performance costs the opera house €20,000. The opera director may send a gift on any night, regardless of the success he has had with gifts on previous nights. The objective is to find a policy for when to send a gift and when not to, that minimizes the total cost from the 5 nights.

- **(a)–5 pt** Formulate the problem as a stochastic dynamic program. What do you choose as stages, states, decisions and optimal value function?
- **(b)–6 pt** What is the recurrence relation of the optimal value function?
- **(c)–6 pt** Use dynamic programming to solve the problem. What are the opera director's expected minimum total costs?
- **(d)–3 pt** What is the optimal policy?

4. (25 points) In each week, a computer manufacturer's product either sells well or it sells poorly. If the product sells well in a given week and the computer manufacturer takes no further action, he earns €6,000 during that week and the product will still sell well in the next week with probability 0.5 (i.e., with probability 0.5, the product will sell poorly in the next week). Alternatively, the computer manufacturer can choose to advertise the product when it sells well. Advertising costs €2,000, but increases the probability that the product will sell well during the next week to 0.8. If the product sells poorly in a given week and the computer manufacturer takes no further action, he 'earns' −€3,000, and the product will sell well in the next week with probability 0.4. Alternatively, the computer manufacturer can choose to do research to improve his product when it sells poorly. Doing research costs €2,000, but increases the probability that the product will sell well during the next week to 0.7. The computer manufacturer aims to maximize his long-term discounted earnings, using a weekly discount factor of 0.9.

(a)–4 pt Model this problem as a Markov decision problem. What do you choose as states, decisions and optimal value function?

(b)–4 pt Determine the direct rewards and transition probabilities for each state and decision.

(c)–4 pt Formulate the corresponding optimality equations explicitly for this specific problem (i.e., not in generic form).

The computer manufacturer currently advertises when the product sells well, and does research when the product sells poorly.

(d)–6 pt Use the policy iteration algorithm to determine whether or not the computer manufacturer's current policy is optimal.

(e)–3 pt Formulate the linear program that the computer manufacturer could have used to determine the optimal policy.

(f)–4 pt In the optimal solution to this linear program, which constraints are binding (i.e., have no slack), and what is the objective function value?