Faculty of Electrical Engineering, Mathematics and Computer Science
Applied Finite Elements, Mastermath
Exam, April 23, 2017–2018, Educatorium, Lecture room beta

$$\text{Exam Grade} = \frac{\text{Sum over all credits}}{2}.$$

1 Given the following functional,

$$F[u] = \int_{x_1}^{x_2} g(x,u(x))\sqrt{1+(u'(x))^2}\,dx.$$

where $g = g(x,u)$ is a smooth function. We are interested in the minimiser for the above functional:

Find u, subject to the constraints $u(x_1) = u_1$ and $u(x_2) = u_2$ where $x_1 < x_2$, such that $F(u) \leq F(v)$ for all v subject to $v(x_1) = u_1$ and $v(x_2) = u_2$.

- a Derive the Euler-Lagrange equation. (3 pt)
- b Derive the Ritz equations. (2 pt)

2 Given the following boundary value problem in domain Ω with boundary $\partial\Omega$

$$\begin{aligned} &-\nabla\cdot(k(x,y)\nabla u) + \mathbf{v}\cdot\nabla u = f(x,y), && \text{in } \Omega, \\ &u = 0, && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $k(x,y)$ and $f(x,y)$ are given functions with $k(x,y) > 0$, and $\mathbf{v}$ is a given vector.

- a Why can we not write this problem into a minimisation problem if $\mathbf{v} \neq \mathbf{0}$? (2 pt)
- b Take $\mathbf{v} = \mathbf{0}$. Prove that the above problem can be written into a minimisation problem. (3 pt)
- c Give the minimisation problem in which the square of ∇u and the square of u are integrable over Ω (that is $u \in H^1(\Omega)$). (2 pt)

3 We consider the following boundary value problem for $u = u(x,y)$ to be determined in $\Omega \subset \mathbb{R}^2$ (bounded by $\partial\Omega$):

$$\begin{cases} -\nabla\cdot(D(u)\nabla u) + \mathbf{v}\cdot\nabla u = f(x,y), & \text{in } \Omega, \\ u = g(x,y), & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $D(u)$ is a positive function of u, $\mathbf{v}$ is a given vector, and $f(x,y)$ and $g(x,y)$ are given functions.

- a Derive the weak formulation in which the order of spatial derivatives is minimized. (2 pt)
- b Derive the Galerkin Equations to the weak form in part a. (1 pt)
- c We use Picard's Fixed Point Method to solve the resulting nonlinear problem. Describe how you would approximate the solution using successive approximations. (2 pt)
- d We use linear triangular elements to solve the problem. All answers may be expressed in terms of $|\Delta_e|$, being twice the area of element e, and $\beta_i = \frac{\partial \lambda_i}{\partial x}$ and $\gamma_i = \frac{\partial \lambda_i}{\partial y}$.
  - i Compute the element matrix and element vector for an internal triangle. (2 pt)
  - ii Compute the element matrix and element vector for a boundary element. (1 pt)

  You may use Newton-Cotes' approximation for numerical integration, which reads as

$$\int_e h(x,y)d\Omega = \frac{|\Delta_e|}{6}\sum_{p\in\{1,2,3\}} h(x_p,y_p), \text{ for triangle } e \text{ with vertices } (x_p,y_p).$$