Faculty of Electrical Engineering, Mathematics and Computer Science
Applied Finite Elements, Mastermath
EXAM MAY 28, 2018: 13:30 – 16:30 O'Clock @ Minnaert Building,
Room MIN 0.11, Leuvenlaan 4, Utrecht University

1 We consider the following boundary value problem for $u = u(x,y)$ to be determined in $\Omega \subset \mathbb{R}^2$ (bounded by $\partial\Omega$) :

$$\begin{cases} \nabla \cdot [\mathbf{v}(x,y)u - D(x,y)\nabla u] = f(x,y), & \text{in } \Omega, \\ \mathbf{v}(x,y) \cdot \mathbf{n}u - D(x,y)\dfrac{\partial u}{\partial n} = g(x,y), & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Here $\mathbf{v}(x,y)$, $f(x,y)$, $g(x,y)$ are given functions and $D = D(x,y) > 0$ in Ω.

a Derive the compatibility condition for f and g. (1 pt)

b Derive the weak formulation in which the order of spatial derivatives is minimized.
Hint: Keep the terms between the brackets as one expression. (2 pt)

c Derive the Galerkin Equations to the weak form in part b. (1 pt)

d We use linear triangular elements to solve the problem. All answers may be expressed in terms of the coefficients in the equations and in the coefficients in $\phi_i = \alpha_i + \beta_i x + \gamma_i y$ as well as in $|\Delta|$ being twice the area of element e.

i Demonstrate the following formula for Newton-Cotes' approximation for numerical integration:

$$\int_e h(x,y)d\Omega = \frac{|\Delta_e|}{6} \sum_{p\in\{1,2,3\}} h(x_p,y_p), \text{ for triangle } e \text{ with vertices } (x_p,y_p).$$

Compute the element matrix and element vector for an internal triangle. (3 pt)

ii Compute the element matrix and element vector for a boundary element. Hint: Use Newton–Cotes integration. (1 pt)

2 We consider bi-linear quadrilateral elements to solve a finite-element problem. In the element we define bi-linear basis functions $\phi_i(\mathbf{x})$ corresponding to the 4 vertices, with coordinates $\mathbf{x}_p$, $p \in \{1,2,3,4\}$, which are used to approximate the solution. The basis-functions are defined through $\phi_i(\mathbf{x}_j) = \delta_{ij}$.

a Explain why we cannot use the form $\phi_i(x,y) = a_i + b_i x + c_i y + d_i xy$ (*Hint:* one may consider the case $\mathbf{x}_1 = (1,0)$, $\mathbf{x}_2 = (0,1)$, $\mathbf{x}_3 = (-1,0)$ and $\mathbf{x}_4 = (0,-1)$). (2 pt)

b We use an iso-parametric transformation, which is defined by

$$(T) : \mathbf{x}(s,t) = \mathbf{x}_1(1-s)(1-t) + \mathbf{x}_2 s(1-t) + \mathbf{x}_3 st + \mathbf{x}_4(1-s)t, \text{ with } s,t \in [0,1],$$

to map the quadrilateral onto a reference square.

i Consider the reference element $\tilde{e} = (0,1)\times(0,1)$, show that the Newton–Cotes Rule applied to a unit square, $\tilde{e}$, is given by

$$\int_{\tilde{e}} g(s,t)d\Omega = \frac{1}{4}\left(g(0,0) + g(1,0) + g(1,1) + g(0,1)\right). \tag{2}$$

(2 pt)

ii The transformation (T) can be rearranged into

$$\mathbf{x}(s,t) = \mathbf{x}_1 + (\mathbf{x}_2 - \mathbf{x}_1)s + (\mathbf{x}_4 - \mathbf{x}_1)t + (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_4)st.$$

Compute the Jacobian matrix of the transformation, $\frac{\partial(x,y)}{\partial(s,t)}$ and show that the determinant of the Jacobian of the transformation is given by

$$\det(\frac{\partial(x,y)}{\partial(s,t)}) = (x_2 - x_1 + A_x t)(y_4 - y_1 + A_y s) - (y_2 - y_1 + A_y t)(x_4 - x_1 + A_x s),$$

with $A_x = x_1 - x_2 + x_3 - x_4$ and $A_y = y_1 - y_2 + y_3 - y_4$, and compute the determinant of the Jacobian on the four vertices that were given in assignment 1a. (2 pt)

iii Compute the Jacobian matrix of the inverse transformation. Evaluate the inverse Jacobian matrix on the vertices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ given in assignment 1a. (2 pt)

c i Let ϕ_1, ϕ_2, ϕ_3 and ϕ_4 be the bilinear basis-functions over quadrilateral element e. Show that $\nabla\phi_i = \left(\frac{\partial(s,t)}{\partial(x,y)}\right)^T \nabla_{st}\phi_i$, where $\nabla_{st}\phi_i$ is the gradient vector with partial derivatives of ϕ_i with respect to s and t. (1 pt)

ii Use the Newton-Cotes Rule to compute $S_{11}^e = \int_e ||\nabla\phi_1||^2 d\Omega$, where e is a quadrilateral with vertices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ that were defined in assignment 1a. (2 pt)

d What is the size of the element matrix for an internal quadrilateral element if a single partial differential equation is solved? Motivate the answer. (1 pt)

$$\text{Written Exam Grade} = \max(\frac{\text{Sum over all credits}}{2}, 1).$$