Examination

Advanced Discrete Optimization
Spring Semester 2026

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Information

General Information

- The examination lasts 180 minutes.
- The grade is determined as $\max\{1, \text{obtained points}\}$.
- Switch off your mobile phone any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- Part 1 has 3 questions and part 2 has 3 questions.
- Answers have to be provided in English.
- All your answers should be clearly written down and provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given in the first line of the respective question.
- Give your answers to the two parts on separate sheets!!!

GOOD LUCK

Part 1

Consider the *Set Multicover* problem defined as follows:

- **Given:**
  - Ground set of elements $E = \{e_1, e_2, \dots e_n\}$, each element e_i associated with an integer *requirement* $r_i \geq 1$.
  - Subsets of elements $S_1, S_2, \dots S_m \subseteq E$, each S_j with a weight $w_j \geq 1$.
- **Goal:** Select a multiset I of indices from $\{1, 2, \dots m\}$ minimizing $\sum_{j \in I} w_j$ (counting multiplicities) subject to $|\{j \in I : e_i \in S_j\}| \geq r_i, \forall e_i \in E$ (again, accounting for multiplicities).

Note that when all $r_i = 1$ and if I is required to be a set, would recover the Set Cover problem.

Let $f = \max_i |\{j : e_i \in S_j\}|$.

Exercise 1 0.5+0.5+1 points

1. Formulate Set Multicover as an ILP.
2. Derive the dual of the LP relaxation.
3. Design a primal-dual approximation algorithm for Set Multicover and show that it is a f-approximation algorithm.

Exercise 2 2 points

Consider the *Online Set Multicover* problem, which is defined identically to Set Multicover, except that the elements of E (and the associated requirements) arrive sequentially, and each element needs to be (multi-)covered upon arrival. Upon arrival of an element e_i, the algorithm can only add sets to I and cannot remove any. Design and analyze an online primal-dual algorithm that produces a fractional $O(\log f)$-competitive solution to Online Set Multicover. Assume in this and the next exercise that all requirements r_i are constant.

Exercise 3 1 point

Consider again Online Set Multicover, but assume that you are also given a multiset I' that constitutes a feasible solution (think of it as a predicted optimal solution), and an input parameter $\lambda \in [0, 1]$. Design an algorithm that produces a solution of cost that is for any input instance simultaneously bounded from above by

- $O\left(\frac{1}{1-\lambda}\right)$ times the cost of I', and
- $O\left(\log\left(\frac{f}{\lambda}\right)\right)$ times the cost of an optimal solution for that instance.

You do not need to analyze this algorithm.

Part 2

Exercise 4

1+0.5+0.75 points

A company has a set of customers $C = \{1, \dots, m\}$ and facilities $F = \{1, \dots, n\}$. Facilities can only serve customers when they are open, which creates a cost $K > 0$. Each customer needs to be assigned to at most one open facility, and when customer c is served by facility f, the company gains a revenue of $r_{c,j} > 0$. Moreover, each open facility can serve at most U customers, where $2 \leq U \leq m - 1$. The goal of the company is to decide which facilities to open and which customers to serve to maximize profit.

Consider the following two integer programming formulations of the problem:

$$
\begin{array}{rlll}
\max & \sum_{c \in C} \sum_{f \in F} r_{c,j} x_{c,j} - \sum_{f \in F} K y_f & & \text{(1a)} \\
& \sum_{c \in C} x_{c,f} \leq U y_f, & f \in F, & \text{(1b)} \\
& \sum_{f \in F} x_{c,f} \leq 1, & c \in C, & \text{(1c)} \\
& x_{c,f} \in \{0,1\}, & c \in C,\ f \in F, & \text{(1d)} \\
& y_f \in \{0,1\}, & f \in F & \text{(1e)}
\end{array}
$$

$$
\begin{array}{rlll}
\max & \sum_{c \in C} \sum_{f \in F} r_{c,j} x_{c,j} - \sum_{f \in F} K y_f & & \text{(2a)} \\
& \sum_{c \in C} x_{c,f} \leq U, & f \in F, & \text{(2b)} \\
& \sum_{f \in F} x_{c,f} \leq 1, & c \in C, & \text{(2c)} \\
& x_{c,f} \leq y_f, & c \in C,\ f \in F, & \text{(2d)} \\
& x_{c,f} \in \{0,1\}, & c \in C,\ f \in F, & \text{(2e)} \\
& y_f \in \{0,1\}, & f \in F & \text{(2f)}
\end{array}
$$

Let P and Q be the feasible regions of the LP relaxations of Model (1) and (2), respectively. Moreover, let Q_I be the convex hull of the integer points contained in Q.

(a) Prove that neither $P \subseteq Q$ nor $Q \subseteq P$ holds.

(b) Compute the dimension of Q_I.

(c) Prove that, for every $c \in C$ and $f \in F$, inequality $x_{c,f} \leq y_f$ defines a facet of Q_I.

Exercise 5

0.25+1 points

Consider the knapsack polytope

$$P = \operatorname{conv}\{x \in \{0,1\}^6 : x_1 + 2x_2 + 3x_3 + 3x_4 + 5x_5 + 6x_6 \leq 6\}.$$

(a) List *all* minimal covers of P. You do *not* need to justify your answer.

(b) Consider the minimal cover $C = \{1, 3, 4\}$. Compute the lifted minimal cover inequality for lifting sequence $5, 2, 6$.

Exercise 6 1.5 points

Consider the linear program

$$\begin{aligned}\min x + y_1 + 2y_2 + 3y_3 \\ x + y_1 + y_2 + y_3 &= 10 \\ y_2 + y_3 &= 5 \\ x, y_1, y_2, y_3 &\geq 0.\end{aligned}$$

Suppose we want to solve the linear program using Benders decomposition, where we only keep x in the master problem, i.e., we consider the problem

$$\begin{aligned}\min x + z \\ x, z \geq 0,\end{aligned}$$

where z models the contribution of the y-variables to the objective (note that we have already incorporated that the y-objective cannot be negative).

The optimal solution of this master problem is $(\bar{x}, \bar{z}) = (0, 0)$. Apply **one** iteration of Benders decomposition to find **one** Benders optimality cut that is violated by $(\bar{x}, \bar{z})$.