Solutions Exam Dynamica, 28-1-2005

1. a. $\frac{1}{2}mv_0^2 - 0 = -mg(0-R) \Rightarrow v_0 = \sqrt{2gR}$

Note that the track cannot do work: there is no friction $\Leftrightarrow$ only normal force perpendicular to the direction of movement

1. b. The particle will reach the highest point B only if gravity in B will not exceed the centripetal force; it's velocity v_B now follows from:

$$\cancel{m}g = \frac{\cancel{m}v_B^2}{R} \Rightarrow v_B = \sqrt{gR}$$

This means that the kinetic energy below should be:

$$U_{k,below} = U_{k,B} + \Delta U_{pot} = \frac{1}{2}mv_B^2 + mg(2R) =$$
$$= \frac{1}{2}mgR + 2mgR = \frac{5}{2}mgR$$

1. c

$$U_{spring,before\,release} = U_{k,before\,entering\,loop}$$
$$\frac{1}{2}k\delta^2 = \frac{1}{2}mv_0^2$$

<u>Lower boundary</u>: If the particle comes beyond A it will lose contact; below it won't:

$$\frac{1}{2}k\delta_{min}^2 = \frac{1}{2}mv_0^2 = mgR \Rightarrow \delta_{min} = \sqrt{\frac{2mgR}{k}}$$

<u>Upper boundary</u>: The particle will make the complete loop if it reaches B (see 1b); otherwise it will leave the track. So:

$$\frac{1}{2}k\delta_{max}^2 = \frac{5}{2}mgR \Rightarrow \delta_{max} = \sqrt{\frac{5mgR}{k}}$$

Therefore:

$$\sqrt{\frac{2mgR}{k}} < \delta < \sqrt{\frac{5mgR}{k}}$$

2. a. $M \leq m$. Use formulas 6-24 and 6-25 or momentum + kin. energy conservation

w = after collision 2
u = after collision 1
v = before collision 1

$1 \to 2$: $u_1 = 0$, $u_2 = v_0$ $(u_3 = 0)$

$2 \to 3$: $w_1 = 0$, $w_2 = \frac{m-M}{m+M} v_0$

$$w_3 = \frac{2m}{m+M} v_0$$

if $M \leq m$ then $w_2 = \frac{m-M}{m+M} v_0 \geq 0$, so both the 2nd and the 3rd particle move to the right and will not collide again.

2. b. $M > m$: particle 3: will move as in a.
particle 2: will move in negative direction and collide with particle 1 again.
Since 1 and 2 have the same mass this leads to a switch in velocities;

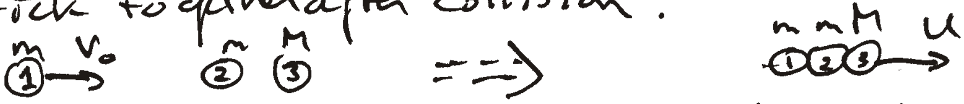
There are three collisions and the final velocities are:

$w_1 = -\left(\frac{M-m}{m+M}\right) v_0$ (moves to left)

$w_2 = 0$

$w_3 = \frac{2m}{m+M} v_0$ (moves to right)

2. c Completely inelastic means: all particles stick together after collision:



momentum conservation: $mv_0 = (2m+M)u \Rightarrow$

$$u = \frac{m}{(2m+M)} v_0$$

Energy loss: $\Delta U_K = \frac{1}{2} m v_0^2 - \frac{1}{2}(2m+M)u^2 =$

$$= \frac{1}{2} m v_0^2 - \frac{1}{2}\cancel{(2m+M)} \frac{m^2 v_0^2}{(2m+M)^{\cancel{2}}} = \frac{1}{2}\left(m - \frac{m^2}{2m+M}\right) v_0^2 = \frac{1}{2}\frac{m(m+M)}{2m+M} v_0^2$$

3 a: $I_{cylinder} = \frac{1}{2} m r^2$ m = mass r = radius

$m = \rho \pi r^2 l$ l = length

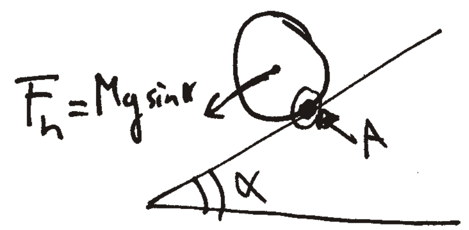
$$I_{tot} = \frac{1}{2}(\rho\pi R^2 L)R^2 - 6\left(\frac{1}{2}(\rho\pi a^2 L)a^2 + (\rho\pi a^2 L)\left(\frac{R}{2}\right)^2\right)$$
$$- \frac{1}{2}(\rho\pi a^2 L)a^2 =$$

$$= \frac{1}{2}\rho\pi L\left(R^4 - 7a^4 - 3a^2R^2\right)$$

$$M = \rho\pi R^2 L - 7\rho\pi a^2 L \Rightarrow \Rightarrow$$
$$\rho\pi L = \frac{M}{R^2 - 7a^2}$$

$$I_{tot} = \frac{1}{2} M \frac{R^4 - 7a^4 - 3a^2R^2}{R^2 - 7a^2}$$

3 b



Determine torque + rotational inertia w.r. to the momentary stationary point A

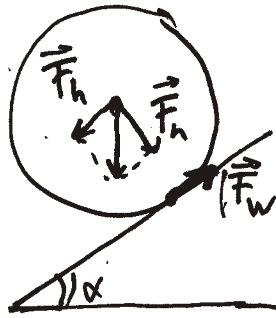
$$I'_{tot} = I_{tot} + MR^2 =$$
$$= \frac{1}{2}M\left(\frac{R^4 - 7a^4 - 3a^2R^2}{R^2 - 7a^2} + 2\frac{R^4 - 7a^2R^2}{R^2 - 7a^2}\right) =$$
$$= \frac{1}{2}M\left(\frac{3R^4 - 7a^4 - 17a^2R^2}{R^2 - 7a^2}\right)$$

$$I'_{tot}\alpha = \frac{I'_{tot}a}{R} = F_h R \Rightarrow a = \frac{F_h R^2}{I'_{tot}} =$$

(α: ang. accelaration)

$$= 2g\sin\alpha \frac{R^4 - 7a^2R^2}{3R^4 - 7a^4 - 17a^2R^2}$$

3 C.



$|\vec{F}_w| = \mu N = \mu F_n$

$$\left.\begin{aligned} F_h &= Mg\sin\alpha \\ F_n &= Mg\cos\alpha \\ F_h - F_w &= Ma \end{aligned}\right\} \Rightarrow$$

$$\mu = \frac{F_w}{N} = \frac{F_w}{F_n} = \frac{F_h - Ma}{F_n} = \frac{Mg\sin\alpha - Ma}{Mg\cos\alpha} =$$

$$= \frac{Mg\sin\alpha - \frac{M^2 g\sin\alpha R^2}{I'_{tot}}}{Mg\cos\alpha} = \tan\alpha\left(1 - \frac{MR^2}{I'_{tot}}\right)$$

$$I'_{tot} = \frac{1}{2}M\left(\frac{3R^4 - 7a^4 - 17a^2R^2}{R^2 - 7a^2}\right) =$$

$$= \frac{1}{2}M\,\frac{1.966\cdot 10^{-3} - 5.670\cdot 10^{-6} - 3.917\cdot 10^{-4}}{2.56\cdot 10^{-2} - 6.3\cdot 10^{-3}} =$$

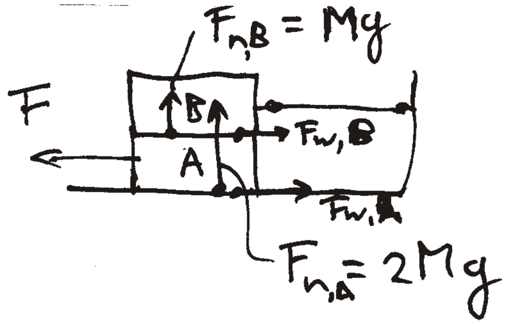
$$= \frac{1}{2}M\,\frac{1.569\cdot 10^{-3}}{1.93\cdot 10^{-2}} = 4.06\cdot 10^{-2}M$$

$$\frac{MR^2}{I'_{tot}} = \frac{R^2\cancel{M}}{4.06\cdot 10^{-2}\cancel{M}} = \frac{2.56\cdot 10^{-2}}{4.06\cdot 10^{-2}} = 0.6305$$

$$\mu = \tan 20^\circ\left(1 - \frac{MR^2}{I'_{tot}}\right) = 0.364(1 - 0.6305) =$$

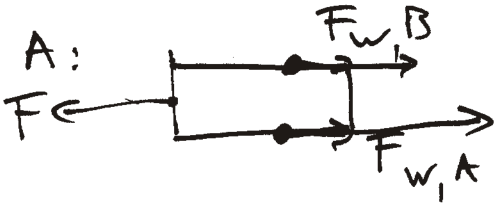
$$= 0.134$$

4 a.



$$F_{w,A} = \mu \cdot F_{n,A} = 2\mu Mg$$
$$F_{w,B} = \mu \cdot F_{n,B} = \mu Mg$$

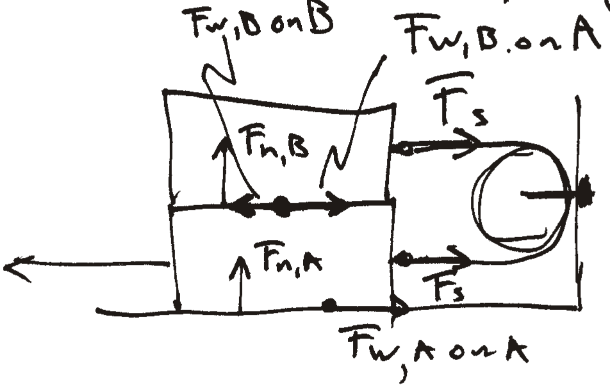
Forces on A:



(in direction of movement)

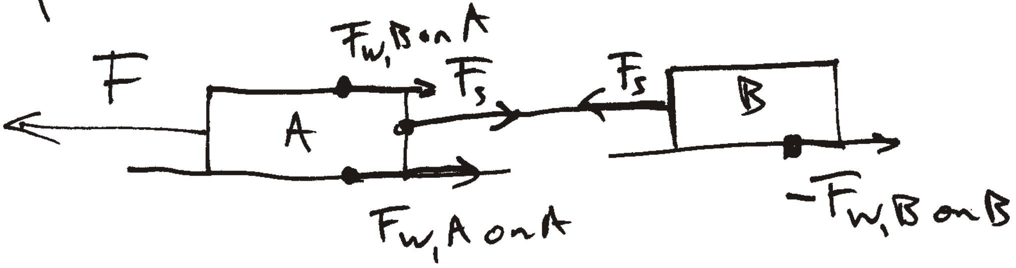
$$Ma = \sum F = F - F_{w,A} - F_{w,B} =$$
$$= F - 3\mu Mg \implies a = \frac{F - 3\mu Mg}{M}$$

4 b.



$$F_{w,B \text{ on } B} = -F_{w,B \text{ on } A}$$
$$= -\mu Mg$$
$$F_{w,B \text{ on } A} = \mu Mg$$
$$F_{w,A \text{ on } A} = 2\mu Mg$$

forces on the system as a whole (fold out!)



$$(M_A + M_B)\, a = F - \left(F_{w,B \text{ on } A} + F_{w,A \text{ on } A} + (-F_{w,B \text{ on } B})\right) =$$
$$= F - (\mu Mg + 2\mu Mg + \mu Mg) \implies$$
$$2Ma = -4\mu Mg + F \implies$$
$$a = \frac{F - 4\mu Mg}{2M}$$

5. a. COM: of disk: $(r_d + l_2)$

of rod: $l_2/2$.

$$r_{COM} = \frac{(r_d + l_{rod}) m_d + \frac{l_2}{2} \cdot m_2}{m_d + m_2}$$

$$= \frac{(0.524 + 0.103)\, 488 + \frac{0.524}{2} \cdot 272}{488 + 272} =$$

$$= 49.6 \text{ cm}$$

5. b. $I_{disk} = \frac{1}{2} m_d r_d^2$ (around center)

$I'_{disk} = \frac{1}{2} m_d r_d^2 + m_d (r_d + l_2)^2$ (around P)

$= \frac{1}{2} m_d r_d^2 + m_d r_d^2 + 2 m_d r_d l_2 + m_d l_2^2$

$I_{rod} = \frac{1}{12} m_2 l_2^2$ (around center)

$I'_{rod} = \frac{1}{12} m_2 l_2^2 + m_2 (l_2/2)^2$ (around P)

$= \frac{1}{3} m_2 l_2^2$

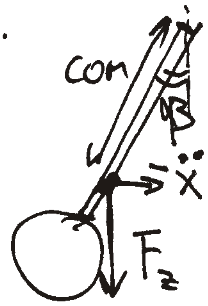
$I_{pendulum} = I'_{disk} + I'_{rod} =$ (around P)

$= \frac{3}{2} m_d r_d^2 + 2 m_d r_d l_2 + m_d l_2^2 + \frac{1}{3} m_2 l_2^2 =$

$= \frac{3}{2} m_d r_d^2 + 2 m_d r_d l_2 + (m_d + \frac{1}{3} m_2) l_2^2 =$

$= 0.2193 \text{ kg m}^2$

5. C.



$$\tau_{F_z} = r_{com} \cdot F_z \cdot \sin\beta \quad \left.\right\} \quad \tau_{F_z} = I\alpha \Longrightarrow$$

$$I\alpha_{pendulum} = I_{pendulum} \cdot \frac{-\ddot{x}}{r_{com}}$$

$$\ddot{x} = \frac{-r_{com}^2 \cdot (m_2 + m_d) g \sin\beta}{I_{pendulum}} =$$

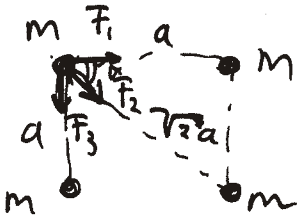
$$= -\frac{r_{com}(m_2 + m_d) g}{I_{pendulum}} x$$

$$\Rightarrow \quad \omega = \sqrt{\frac{r_{com}(m_2 + m_d) g}{I_{pendulum}}} = \frac{2\pi}{T} \Rightarrow$$

$$T = 2\pi \sqrt{\frac{I_{pendulum}}{r_{com}(m_2 + m_d) g}} =$$

$$= 2\pi \sqrt{\frac{0.219 \, kg\,m^2}{0.496\,m \; 0.760\,kg \cdot 9.81\,m/s^2}} =$$

$$= 2\pi \; 0.243\,s = 1.53 \;\, s.$$

7 a. 

$$F_1 = F_3 = G\frac{m^2}{a^2}$$

$$F_2 = G\frac{m^2}{(\sqrt2\, a)^2} = G\frac{m^2}{2a^2}$$

$$F_{tot} = F_2 + 2F_1 \cos 45^\circ =$$

$$= G\frac{m^2}{2a^2} + \cancel{2}\, G\frac{m^2}{a^2}\,\cancel{\tfrac12}\sqrt2 =$$

$$= \left(\tfrac12 + \sqrt2\right) G\frac{m^2}{a^2}$$

F_{tot} should be balanced by the centrifugal force $\Big\} \Rightarrow$

$$F_c = m\omega^2 r \overset{r=\frac12\sqrt2\, a}{=} \tfrac12\sqrt2\, m\omega^2 a$$

$$\left(\tfrac12+\sqrt2\right) G m^{\cancel{2}}/a^2 = \tfrac12\sqrt2\,\cancel{m}\,\omega^2 a \Rightarrow$$

$$\omega^2 = \sqrt2\left(\tfrac12+\sqrt2\right) G\frac{m}{a^3} = \left(\tfrac12\sqrt2+2\right) G\frac{m}{a^3} \Rightarrow$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac12\sqrt2+2\right) G\frac{m}{a^3}}}$$

7 b. Now $F_{tot} = \left(\tfrac12+\sqrt2\right) G\frac{m^2}{a^2} + G\frac{Mm}{\left(\frac12\sqrt2\right)^2 a^2} =$ $\boxed{\omega' = 5\omega}$

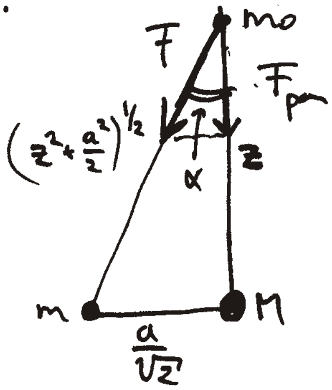
$$= \left(\tfrac12+\sqrt2\right) G\frac{m^2}{a^2} + 2G\frac{Mm}{a^2} \Rightarrow$$

$$\underbrace{\tfrac12\sqrt2\,\cancel{m}\,\omega'^2 a}_{\frac{25}{2}\sqrt2\,\cancel{m}\,\omega^2 a} = \underbrace{\left(\tfrac12+\sqrt2\right) G\frac{m^{\cancel{2}}}{a^2}}_{\frac12\sqrt2\,\cancel{m}\,\omega^2 a} + 2G\frac{M\cancel{m}}{a^2} \Rightarrow$$

$$M = \frac{a^3}{2G}\left(\tfrac{25}{2}\sqrt2 - \tfrac12\sqrt2\right)\omega^2 = 6\sqrt2\,\frac{a^3}{G}\,\omega^2 =$$

$$= 6\sqrt2\left(\tfrac12\sqrt2+2\right)\frac{\cancel{a^3}}{\cancel{G}}\,\cancel{G}\frac{m}{\cancel{a^3}} = \left(6+12\sqrt2\right) m$$

7 c.



$$F = G\frac{m_0 m}{z^2+\frac{a^2}{2}}$$

$$F_{par} = G\frac{m_0 m}{z^2+\frac{a^2}{2}}\cos\alpha = G\frac{m_0 m z}{(z^2+\frac{a^2}{2})^{3/2}}$$

$$F_{tot} = G\frac{m_0 M}{z^2} + 4G\frac{m_0 m z}{(z^2+a^2/2)^{3/2}}$$

$$(= F_{centralmass} + 4F_{par})$$

7 d. Change in kinetic energy:

$$\Delta U_k = -\int_{z_0}^{R_0} F_{tot}\,dz = +\int_{R_0}^{z_0} G\frac{m_0 M}{z^2}\,dz$$

$$+ 4Gm_0 m\int_{R_0}^{z_0}\frac{z\,dz}{(z^2+a^2/2)^{3/2}} \underset{\substack{\text{int}\\ \text{for } q=3/2}}{=} -G\frac{m_0 M}{z}\Big|_{R_0}^{z_0}$$

$$-4Gm_0 m\frac{1}{2\cdot\frac{1}{2}}\cdot\frac{1}{(z^2+a^2/2)^{1/2}}\Big|_{R_0}^{z_0} = Gm_0 M\left(\frac{1}{R_0}-\frac{1}{z_0}\right)$$

$$+ 4Gm_0 m\left(\frac{1}{\sqrt{R_0^2+a^2/2}} - \frac{1}{\sqrt{z_0^2+a^2/2}}\right) = \frac{1}{2}m_0 v^2$$

$$\Rightarrow v = \sqrt{2G}\left(\sqrt{M\left(\frac{1}{R_0}-\frac{1}{z_0}\right) + 4m\left(\frac{1}{\sqrt{R_0^2+a^2/2}} - \frac{1}{\sqrt{z_0^2+a^2/2}}\right)}\right)$$

$$\Bigg(= \sqrt{2Gm}\sqrt{\left(6+\frac{12}{\sqrt{2}}\right)\left(\frac{1}{R_0}-\frac{1}{z_0}\right) + 4\left(\frac{1}{\sqrt{R_0^2+a^2/2}} - \frac{1}{\sqrt{z_0^2+a^2/2}}\right)}$$

$$= 2\sqrt{2Gm}\sqrt{\left(\frac{3}{2}+\frac{3}{\sqrt{2}}\right)\left(\frac{1}{R_0}-\frac{1}{z_0}\right) + \left(\frac{1}{\sqrt{R_0^2+a^2/2}} - \frac{1}{\sqrt{z_0^2+a^2/2}}\right)}\Bigg)$$