1 Geg $P(M) = 0{,}9$ $P(C|M) = 0{,}8$ $P(C|\bar{M}) = 0{,}1$

a. $P(\bar{M}\bar{C}) = P(\bar{C}|\bar{M})P(\bar{M}) = 0.9 \cdot 0.1 = 0{,}09$

b. $P(\bar{M}|C) = P(C|\bar{M})P(\bar{M}) \,/\, P(C)$

$$= \frac{P(C|\bar{M})P(\bar{M})}{P(C|\bar{M})P(\bar{M}) + P(C|M)P(M)} = \frac{1}{73}$$

2 a

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = e^{-x}\int_0^{\infty} x e^{-xy}\,dy = e^{-x}, \quad x \geq 0$$

b.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = x e^{-xy}, \quad y \geq 0$$

c. $F_Z(z) = P(Z \leq z) = P(XY \leq z)$

$$= \iint_{\{(x,y):\, xy \leq z\}} f(x,y)\,dy\,dx$$

$$= \int_0^{\infty}\int_0^{z/x} x e^{-x(y+1)}\,dy\,dx$$

$$= \int_0^{\infty} e^{-x}\int_0^{z/x} x e^{-xy}\,dy\,dx$$

$$= \int_0^{\infty} e^{-x}(1-e^{-z})\,dx = 1-e^{-z}, \quad z \geq 0$$

$$f_Z(z) = e^{-z}, \quad z \geq 0$$

3 a $P(X>1) = 1 - (1-p)^n - np(1-p)^{n-1}$

and so

$$\phi''(0) = E[X^2]$$

In general, the nth derivative of $\phi(t)$ evaluated at $t = 0$ equals $E[X^n]$, that is,

$$\phi^n(0) = E[X^n], \qquad n \geqslant 1$$

We now compute $\phi(t)$ for some common distributions.

Example 2.39 (The Binomial Distribution with Parameters n and p)

b

$$\begin{aligned}\phi(t) &= E[e^{tX}] \\ &= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\ &= (pe^t + 1 - p)^n\end{aligned}$$

Hence,

c

$$\phi'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

and so

$$E[X] = \phi'(0) = np$$

which checks with the result obtained in Example 2.17. Differentiating a second time yields

$$\phi''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

and so

$$E[X^2] = \phi''(0) = n(n-1)p^2 + np$$

Thus, the variance of X is given

$$\begin{aligned}\mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= np(1-p) \quad \blacksquare\end{aligned}$$

4 a $X_i \sim N(75, 100)$

$E\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} EX_i = 75$

2 $\operatorname{var} \bar{X}_n = \frac{1}{n^2} \sum_{i=1}^{n} \operatorname{var} X_i = \frac{100}{n}$

$\therefore \bar{X}_n \sim N(75, \frac{100}{n})$

b $P(\sum_{i=1}^{12} X_i > 1000) = P(\bar{X}_{12} > \frac{1000}{12})$

$= P\left(\frac{\bar{X}_{12} - 75}{10/\sqrt{12}} > \frac{1000/12 - 75}{10/\sqrt{12}} \right)$

2 $= 1 - \Phi\left(\frac{5}{\sqrt{3}} \right) = 1 - \Phi(2.886) = 0{,}0019$

c. $EX_1 = \frac{1}{2}(a+b) = 75$

$\operatorname{var} X_1 = \frac{1}{12}(b-a)^2 = 100$

2 $\therefore a = 75 - 10\sqrt{3} \quad b = 75 + 10\sqrt{3}$

d $P(\sum_{i=1}^{n} X_i > 1000) < 0{,}001$

$\Leftrightarrow P\left(\frac{\sum_{i=1}^{n} X_i - 75n}{10\sqrt{n}} > \frac{1000 - 75n}{10\sqrt{n}} \right) < 0{,}001$

2 $\overset{\text{c.L.S.}}{\sim} 1 - \Phi\left(\frac{1000 - 75n}{10\sqrt{n}} \right) < 0{,}001$

$\Leftrightarrow \Phi(\ldots) > 0{,}999$ 1

$\Leftrightarrow \frac{1000 - 75n}{10\sqrt{n}} \geq 3{,}10$

$\Leftrightarrow 31\sqrt{n} + 75n \leq 1000 \Leftrightarrow n \leq 11$

5. a $P(Y=k) = \int_0^\infty P(Y=k \mid X=x)\lambda e^{-\lambda x}\,dx$ 2

$= \lambda \int_0^\infty \frac{(\mu x)^k}{k!} e^{-(\lambda+\mu)x}\,dx$

3 $P(Y=0) = \frac{\lambda}{\lambda+\mu}$

$P(Y=k) = \frac{\mu}{\lambda+\mu} P(Y=k-1)$

$\therefore\ P(Y=k) = \left(\frac{\mu}{\lambda+\mu}\right)^k \frac{\lambda}{\lambda+\mu} \qquad k=0,1,\ldots$

b

1 $E(Y \mid X=x) = \sum_{k=0}^\infty k\, P(Y=k \mid X=x) = \mu x$

c

2 $EY = E(E(Y \mid X)) = E(\mu X) = \mu EX = \frac{\mu}{\lambda}$

d

$P(X \le x \mid Y=0)$

2 $= \frac{1}{P(Y=0)} P(X \le x, Y=0) = \frac{1}{P(Y=0)} \int_0^x P(Y=0 \mid X=u)\lambda e^{-\lambda u}\,du$

$= \frac{1}{P(Y=0)} \int_0^x \lambda e^{-(\lambda+\mu)u}\,du = \frac{1}{P(Y=0)} \frac{\lambda}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)x}\right), \ x \ge 0$

$= 1 - e^{-(\lambda+\mu)x}, \ x \ge 0$

$f_{X|Y}(x \mid 0) =$

$= (\lambda+\mu)e^{-(\lambda+\mu)x}, \quad x \ge 0$

6

a $P(V > t) = P(X > t, Y > t) = e^{-\left(\frac{1}{a}+\frac{1}{b}\right)t}, \; t \geq 0$

$\therefore \; V \sim E\left(\frac{1}{a}+\frac{1}{b}\right)$

$\therefore \; EV = \frac{ab}{a+b} \qquad \text{var}\, V = \left(\frac{ab}{a+b}\right)^2$

b $P(W = X) = P(X > Y) = \int_0^\infty P(X > Y \mid Y = y)\, \frac{1}{b} e^{-\frac{1}{b}y}\, dy$

$= \ldots = \frac{a}{a+b}$

c $V + W = X + Y$

$\therefore \; EV + EW = EX + EY = a + b$

$\therefore \; EW = a + b - \frac{ab}{a+b}$