Lineaire Structuren - 1

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1. (VS 3) is niet geldig.

Neem (a_1, a_2) met $a_2 \neq 0$

Dan is er geen vector (b_1, b_2) zodat

$$(a_1, a_2) + (b_1, b_2) = (a_1, a_2)$$

$\Rightarrow$ V is géén vectorruimte.

2.

$$a_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + a_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + a_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + a_4\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a_1 + 2a_4 = 0$

$a_1 + a_2 - a_4 = 0$

$a_1 + a_2 + a_3 + a_4 = 0$

$a_1 + a_2 + a_3 + 3a_4 = 0$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$a_4 = 0$

$a_3 + 2a_4 = 0 \Rightarrow a_3 = 0$

$a_2 - 3a_4 = 0 \Rightarrow a_2 = 0$

$a_1 + 2a_4 = 0 \Rightarrow a_1 = 0$

Alléén triviale oplossing $\Rightarrow$ S lin. onafhankelijk

3. a) Zij V-vectorruimte, $\beta = \{u_1, u_2, \dots, u_n\}$ geordende basis. De coördinaatvector van $x \in V$ is de vector $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ zodat

$$x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$$

b) Stel voor dat

$$x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$$

en $$x = b_1 u_1 + b_2 u_2 + \dots + b_n u_n$$

$$x - x = \underline{0} = (a_1 - b_1) u_1 + (a_2 - b_2) u_2 + \dots + (a_n - b_n) u_n$$

$\{u_1, u_2, \dots, u_n\}$ - lin. onafh. $\Rightarrow$

$$a_1 - b_1 = a_2 - b_2 = \dots = a_n - b_n = 0$$

$\Rightarrow a_1 = b_1, \; a_2 = b_2, \dots, a_n = b_n$

$\Rightarrow$ de coördinaatvector is uniek. $\blacksquare$

4. a) 1) $T(f(x) + g(x)) = 2(f(x) + g(x)) + (f(x) + g(x))' + (f(x) + g(x))''$

$$= 2f(x) + 2g(x) + f'(x) + g'(x) + f''(x) + g''(x)$$
$$= \left(2f(x) + f'(x) + f''(x)\right) + \left(2g(x) + g'(x) + g''(x)\right)$$
$$= T(f(x)) + T(g(x))$$

voor alle $f, g \in P_2(\mathbb{R})$

2) Verder, $\forall \; f \in P_2(\mathbb{R}), \; c \in \mathbb{R}$

$$T(cf(x)) = 2cf(x) + (cf(x))' + (cf(x))''$$
$$= c\,(2f(x) + f'(x) + f''(x))$$
$$= c\;T(f(x))$$

1) + 2) $\Rightarrow$ T is een lin. afbeelding.

b) $T(x^2) = 2x^2 + 2x + 2$

$T(x) = 2x + 1$

$T(1) = 2$

$$[T]_\beta = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

c) $f(x) = x^2 + 3x + 4$ $\qquad [f(x)]_\beta = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

$$[T]_\beta \, [f(x)]_\beta = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = $$

$$= \begin{pmatrix} 13 \\ 8 \\ 2 \end{pmatrix}$$

$$[T(f(x))]_\beta = [\underbrace{2x^2 + 6x + 8}_{2f(x)} + \underbrace{2x + 3}_{f'(x)} + \underbrace{2}_{f''(x)}]_\beta$$

$$= [2x^2 + 8x + 13]_\beta = \begin{bmatrix} 13 \\ 8 \\ 2 \end{bmatrix}$$

d) $\text{rang}\,[T]_\beta = 3 \Rightarrow [T]_\beta$ inverteerbaar

$\Rightarrow$ T is een isomorfisme.

(Andere oplossingen mogelijk)

5. $T: V \to W$ $\quad T$ one-to-one

$\Rightarrow \dim N(T) = 0$

Dimensistelling:

$$\dim(V) = \text{rang}(T) + \dim N(T) = \text{rang}(T)$$
$$= \dim(R(T)).$$

$R(T)$ is een deelruimte van $W \Rightarrow \dim(R(T)) \leq \dim(W)$

$\Rightarrow \dim(V) \leq \dim(W)$ $\blacksquare$

6. a) $\begin{pmatrix} 1 & 2 & -1 & 3 & 0 & 1 \\ 2 & 2 & 0 & 3 & 2 & 4 \\ -1 & 4 & -5 & 9 & -6 & -6 \end{pmatrix} \to$

$\to \begin{pmatrix} 1 & 2 & -1 & 3 & 0 & 1 \\ 0 & -2 & 2 & -3 & 2 & 2 \\ 0 & 6 & -6 & 12 & -6 & -5 \end{pmatrix} \to$

$\to \begin{pmatrix} 1 & 2 & -1 & 3 & 0 & 1 \\ 0 & -2 & 2 & -3 & 2 & 2 \\ 0 & 0 & 0 & 3 & 0 & 1 \end{pmatrix} \to$

$\to \begin{pmatrix} 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1/3 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 & -1 & -3/2 \\ 0 & 0 & 0 & 1 & 0 & 1/3 \end{pmatrix}$

x_5 vrij $\quad x_4 = 1/3$

x_3 vrij $\quad x_2 = -3/2 + x_5 + x_3$

$x_1 = 3 - 2x_5 - x_3 \qquad x_3 = s,\ x_5 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ -3/2 \\ 0 \\ 1/3 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

b) $K_H = \left\{ s \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$

$\dim(K_H) = 2$, basis voor K_H is: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

c) rang $A = 5 - \dim(K_H) = 5 - 2 = 3$

$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix} \right\}$ - lin. onafh. kolommen van A.

6 d)

$$B = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

AB is 3×3

7 a) $A^{-1}A = I$

$$\det(A^{-1}A) = \underline{\det(A^{-1})\det(A)} = \det(I) = \underline{1}$$

$$\det(A^{-1}) = (\det(A))^{-1}$$

($\det(A) \neq 0$ want A is inverteerbaar).

b)

$$\det(A) = \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & \alpha \\ 1 & 0 & 3 & 7 \\ -2 & 1 & 2 & 0 \end{pmatrix} =$$

$$= \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & \alpha+8 \\ 1 & 0 & 3 & 7 \\ -2 & 1 & 2 & 0 \end{pmatrix} =$$

$$= (-1)^{2+4}(\alpha+8)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ -2 & 1 & 2 \end{vmatrix} =$$

$$= (\alpha+8)\left((-1)^{2+1} 1 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + (-1)^{2+3} \cdot 3 \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}\right)$$

$$= (\alpha+8)(-1(4-3) - 3(1+4))$$

$$= (\alpha+8)(-4+3-3-12) = -16(\alpha+8)$$

$\det(A) = 0$ als $\alpha = -8$.

In dit geval zijn rijen 1 en 2 lin. afhankelijk.