Uitwerking Tentamen Calculus II d.d. 15-03-1999

1a f van klasse C^2

$f(a,b)=0$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 2e^{u_1} & v_1 \\ -6 & 1 \end{bmatrix} \Rightarrow \det\Big|_{(0,1,3,2)} = \begin{vmatrix} 2 & 3 \\ -6 & 1 \end{vmatrix} = 20 \neq 0 \Rightarrow C^2\text{-afb}$$

b $$\frac{\partial f}{\partial v} = \begin{bmatrix} u_2 & -4 \\ 1 & -2 \end{bmatrix} \Rightarrow (a,b) \Rightarrow \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow D_b U = -\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix} = -\begin{bmatrix} 1/20 & -3/20 \\ 3/10 & 1/10 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/10 & -1/10 \\ -2/5 & 7/5 \end{bmatrix}$$

$$\frac{\partial U_1}{\partial v_1}(b) = \frac{1}{10}$$

c

$$2e^{u_1} + u_2 v_1 - 4v_2 + 3 = 0$$
$$-6u_1 + u_2 + v_1 - 2v_2 = 0$$

p.dif. naar v_1

$$2e^{u_1}\frac{\partial u_1}{\partial v_1} + v_1\frac{\partial u_2}{\partial v_1} + u_2 = 0 \quad \underset{(a,b)}{\Rightarrow} \quad 2\frac{\partial u_1}{\partial v_1} + 3\frac{\partial u_2}{\partial v_1} + 1 = 0$$

$$-6\frac{\partial u_1}{\partial v_1} + \frac{\partial u_2}{\partial v_1} + 1 = 0 \quad \underset{(a,b)}{\Rightarrow} \quad -6\frac{\partial u_1}{\partial v_1} + \frac{\partial u_2}{\partial v_1} + 1 = 0$$

$$\Downarrow$$

$$\frac{\partial u_1}{\partial v_1} = \frac{1}{10} \qquad \frac{\partial u_2}{\partial v_1} = -\frac{2}{5}$$

nogmaals partieel differentieren

$$2e^{u_1}\left(\frac{\partial u_1}{\partial v_1}\right)^2 + 2e^{u_1}\frac{\partial^2 u_1}{\partial v_1^2} + \frac{\partial^2 u_2}{\partial v_1^2}\cdot v_1 + 2\frac{\partial u_2}{\partial v_1} = 0$$

$$-6\frac{\partial^2 u_1}{\partial v_1^2} + \frac{\partial^2 u_2}{\partial v_1^2} = 0$$

(a,b)

$$\Rightarrow 2\frac{\partial^2 u_1}{\partial v_1^2} + 3\frac{\partial^2 u_2}{\partial v_1^2} + \frac{2}{100} - \frac{4}{5} = 0 \qquad 20\frac{\partial^2 u_1}{\partial v_1^2} = \frac{78}{100}$$

$$-6\frac{\partial^2 u_1}{\partial v_1^2} + \frac{\partial^2 u_2}{\partial v_1^2} = 0 \qquad \Rightarrow \frac{\partial^2 u_1}{\partial v_1^2} = \frac{78}{2000}$$

2 3'orde Taylor

$$f(x,y)=\cos(x^2)+\ln\left(\frac{x+1}{y}\right) = \cos(x^2)+\ln(1+x)-\ln(1+(y-1))$$

$$= 1-\frac{x^2}{2}+o(x^3)+x-\frac{x^2}{2}+\frac{x^3}{3}-\left((y-1)-\frac{(y-1)^2}{2}+\frac{(y-1)^3}{3}\right)+o((y-1)^3)$$

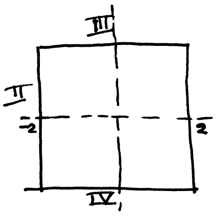
$$= 1-\frac{x^2}{2}+x-\frac{x^2}{2}+\frac{x^3}{3}-(y-1)+\frac{(y-1)^2}{2}-\frac{(y-1)^3}{3}+o(\|(x,y)-(0,1)\|^3)$$

3 a $f_x = 0 = 2x(1+y)^3 \quad f_y = 0 = 3x^2(1+y)^2-12y \Rightarrow (x,y)=(0,0)$!!

b $f_{xx} = 2(1+y)^3 \quad f_{xy} = 6x(1+y)^2 \quad f_{yy} = 6x^2(1+y)-12 \quad (0,0) \Rightarrow \begin{vmatrix} 2 & 0 \\ 0 & -12 \end{vmatrix} < 0$

$\Rightarrow$ zadelpunt

c randextrema



I: $x=2 \quad \bar{f} = 4(1+y)^3-6y^2$

II: $x=-2 \quad \bar{f} = 4(1+y)^3-6y^2$

$\bar{f}_y = 0$

$12(1+y)^2-12y \quad \Rightarrow (-2,-2) \Rightarrow -28$

$=12(1+y+y^2)>0 \quad (+2,-2) \Rightarrow -28$

$(2,2) \Rightarrow 84$

$(-2,2) \Rightarrow 84$

III $y=-2 \quad \tilde{f} = -x^2-24$

$\max(0,-2) = -24 \quad \min(\pm 2,-2) = -28$

IV $y=2 \quad \tilde{f} = 27x^2-24$

$\max(2,2) = 84$

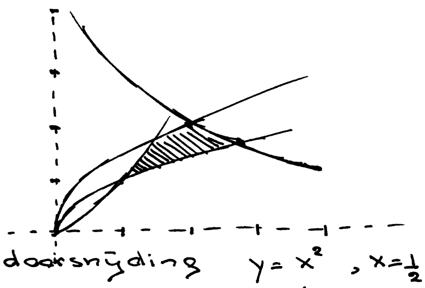
$\min(2,0) = -24$

$\Rightarrow$ globaal max op rand: 84

" min " " : -28

4 G: $1 \leq \frac{x^2}{y} \qquad 1 \leq \frac{y^2}{x} \leq 2 \qquad xy \leq 4$

$\Rightarrow y=x^2, \quad x=y^2, \quad y^2=2x \qquad y=\frac{4}{x}$



doorsnijding $y=x^2, \; x=\frac{1}{2}y^2 \qquad y=\frac{1}{4}y^4 \qquad x=\sqrt[3]{2}, \; y=\sqrt[3]{4}$

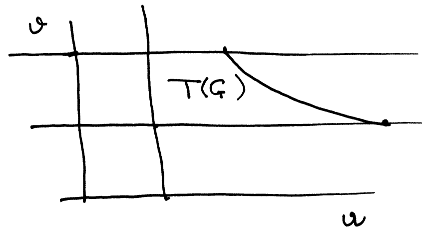
" $y=\frac{4}{x}, \; x=y^2 \qquad y^3=4 \qquad x=2\sqrt[3]{2} \quad y=\sqrt[3]{4}$

" $x=\frac{1}{2}y^2, \; y=\frac{4}{x} \qquad x=2 \quad y=2$

$\Rightarrow$ opsplitsing als herhaalde integralen

b $\displaystyle\int_1^{\sqrt[3]{4}}\Big[\int_{\sqrt{y}}^{y^2} xy\,dx\Big]dy + \int_{\sqrt[3]{4}}^{2}\Big[\int_{\frac{1}{2}y^2}^{\frac{4}{y}} xy\,dx\Big]dy$

c $u = \frac{x^2}{y}$ $\quad v = \frac{y^2}{x}$



$\frac{x^2}{y} = 1 \quad \frac{y^2}{x} = 1 \quad \frac{y^2}{x} = 2 \quad xy = 4$

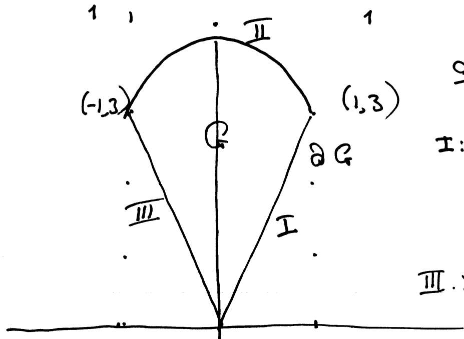
$u = 1 \quad v = 1 \quad v = 2 \quad uv = 4$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3$$

d $\displaystyle\iint_G xy\,dx\,dy = \int\int_{T(G)} uv\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$

$$= \frac{1}{3}\int_1^2\Big[\int_1^{\frac{4}{v}} uv\,du\Big]dv = \frac{1}{3}\int_1^2 \frac{1}{2}u^2v\Big|_{u=1}^{u=\frac{4}{v}} dv = \frac{1}{3}\int_1^2\Big(\frac{8}{v} - \frac{1}{2}v\Big)dv = \frac{8}{3}\ln 2 - \frac{1}{4}$$

5



a $\oint(\underline{\omega}.\underline{n})\,ds = \oint -\omega_2\,dx_1 + \omega_1\,dx_2$

I: $x_1 = t,\ x_2 = 3t$

$$\int_0^1 -t^2dt + 3t.3dt = -\frac{1}{3}t^3 + \frac{9}{2}t^2\Big|_0^1 = -\frac{1}{3} + \frac{9}{2}$$

III. $x_1 = t \quad x_2 = -3t$

$$\int_{-1}^0 -t^2dt + (-3t).-3dt = -\frac{1}{3}t^3 + \frac{9}{2}t^2\Big|_{-1}^{0} = \frac{-1}{3} - \frac{9}{2}$$

$-\mathrm{II} = x_1 = t,\ x_2 = 4 - t^2$

$$\Rightarrow -\int_{-1}^1 -t^2dt + (4-t^2).-2t\,dt = -\int_{-1}^1 -t^2 - 8t + 2t^3dt = -\Big(\frac{-1}{3}t^3 - 4t^2 + \frac{2}{4}t^4\Big)\Big|_{-1}^1 = \frac{2}{3}$$

som van totaal is 0 zie b: $\frac{\partial\omega_1}{\partial x_1} + \frac{\partial\omega_2}{\partial x_2} = 0$ $\square$

6 $S_1: x^2+y^2 \leq 1 \quad z = 4 - x^2 - y^2$
$S_2: x^2+y^2 \leq 1 \quad z = 3\sqrt{x^2+y^2}$ figuur uit vorige opgave geroteerd

b rot w.: $\begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ x^2+y^2, & z^2, & y^2-x^2 \end{vmatrix} = (2y-2z, 2x, -2y)$

div rot w = 0
$\Rightarrow$ antw c = antw b.

parameterisering
$x = r\cos\varphi \qquad \underline{x}_r = (\cos\varphi, \sin\varphi, -2r)$
$y = r\sin\varphi \qquad \underline{x}_\varphi = (-r\sin\varphi, r\cos\varphi, 0)$
$z = 4 - r^2$

$\underline{x}_r \times \underline{x}_\varphi = (2r^2\cos\varphi, 2r^2\sin\varphi, r)$

rot w: $(2r\sin\varphi - 2(4-r^2), 2r\cos\varphi, -2r\sin\varphi)$

$$\Rightarrow \iint_{S_1} (\text{rot}\,\underline{w}.\underline{n})\,dA = \int_0^{2\pi}\int_0^1 4r^3\sin\varphi\cos\varphi - 4r^2(4-r^2)\cos\varphi + 4r^3\sin\varphi\cos\varphi - 2r^2\sin\varphi\, dr\, d\varphi = 0$$

deze opgave kan ook via Stokes behandeld worden

$\partial S_1: \quad x = \cos\varphi, y = \sin\varphi, z = 3$

$$\Rightarrow \int_0^{2\pi} ((\cos^2\varphi + \sin^2\varphi).-\sin\varphi + 3.\cos\varphi + 0)\,d\varphi = 0$$