Oefententamen Analyse II 22 aug 2012.

1a $f_n(x) = \sqrt{x^2 + 1/n^2}$ $\quad x \in [-1,1]$

$f_n(x) \to f(x) = \sqrt{x^2} = |x|$

uniform?

$$|f_n(x) - f(x)| = \left|\sqrt{x^2+1/n^2} - \sqrt{x^2}\right|$$
$$= \left|\left(\sqrt{x^2+1/n^2} - \sqrt{x^2}\right)\left(\frac{\sqrt{x^2+\frac{1}{n^2}}+\sqrt{x^2}}{\sqrt{x^2+1/n^2}+\sqrt{x^2}}\right)\right|$$
$$= \frac{1}{n}\frac{1}{\sqrt{n^2x^2+1}+\sqrt{n^2x^2}}$$
$$\leq \frac{1}{n}$$

Dus geg. $\varepsilon > 0$, $|f_n(x) - f(x)| < \varepsilon$ als $n \geq N = \frac{1}{\varepsilon}$ $\forall x \in [-1,1]$

Dus uniform convergent.

Alternatief

$|f_n(x) - f(x)| = \left|\sqrt{x^2+1/n^2} - \sqrt{x^2}\right|$

kies $h_n(x) = \sqrt{x^2+1/n^2} - \sqrt{x^2}$

$h_n(x) > 0$ en $h_n(0) = \frac{1}{n}$

$h_n'(x) = \frac{x}{\sqrt{x^2+1/n^2}} - \frac{x}{\sqrt{x^2}} = x\left(\frac{1}{\sqrt{x^2+1/n^2}} - \frac{1}{\sqrt{x^2}}\right)$

$= 0$ als $x = 0$

$h_n'(x) < 0$ als $x > 0$

$h_n'(x) > 0$ als $x < 0$

Rand: $h_n(\pm 1) = \sqrt{1+\frac{1}{n^2}} - 1$

$\sqrt{1+\frac{1}{n^2}} - 1 = \sqrt{\frac{n^2+1}{n^2}} - 1 = \frac{1}{n}\left(\sqrt{n^2+1} - n\right) > 0$

en $\frac{1}{n} > \frac{1}{n}\left(\sqrt{n^2+1} - n\right)$ want $1 > \sqrt{n^2+1} - n$

$1 + n > \sqrt{n^2+1}$

$1 + 2n + n^2 > n^2 + 1$

$2n > 0$

Dus het maximum van $h_n(x)$ is

$h_n(0) = \frac{1}{n}$ dus $|h_n(x)| = \sqrt{x^2+\frac{1}{n^2}} - \sqrt{x^2} \leq \frac{1}{n} = \varepsilon$

als $n \geq N = \frac{1}{\varepsilon}$ is unif. conv

1 b $\sum_{n=0}^{\infty} \frac{\sin(\arctan(3^n x^2))}{2^n}$ $\quad x \in [-1, 1]$

$$\left| \frac{\sin(\arctan(3^n x^2))}{2^n} \right| \leq \frac{1}{2^n}$$

Weierstrass: $\sum_{n=0}^{\infty} \frac{1}{2^n} < \infty$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

Reeks conv. abs en unif. op $[-1,1]$

2 a $f(x) = x^4 \arctan\left(\frac{1}{3}x\right)$

$$\arctan\left(\frac{x}{3}\right) = \frac{1}{3}\int_0^x \frac{1}{1+\left(\frac{1}{3}t\right)^2}\, dt$$

$$= \frac{1}{3}\int_0^x \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{3}t\right)^{2k} dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{3}\right)^{2k+1} \int_0^x t^{2k}\, dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{3}\right)^{2k+1} x^{2k+1} \frac{1}{2k+1}$$

Dus $x^4 \arctan\left(\frac{x}{3}\right) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{3}\right)^{2k+1} \frac{1}{2k+1}\, x^{2k+5}$

b verhoudingstest.

$$R = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \text{ met } a_k = (-1)^k \frac{1}{2k+1}\left(\frac{1}{3}\right)^{2k+1} x^{2k+5}$$

$$= \lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{2k+3} \cdot \frac{x^{2k+7}}{3^{2k+3}} \cdot \frac{2k+1}{(-1)^k} \cdot \frac{3^{2k+1}}{x^{2k+5}} \right|$$

$$= \lim_{k\to\infty} \frac{x^2}{9} \frac{2k+1}{2k+3} = \frac{x^2}{9}$$

$R < 1 \Rightarrow \frac{x^2}{9} < 1 \Rightarrow -3 < x < 3 \quad \underline{R = 3}.$

$$f(3) = 81 \sum_{h=0}^{\infty} \frac{(-1)^h}{2h+1}$$

alternerende reeks $\frac{1}{2h+1} \to 0$ als $h \to \infty$: convergeert.

$$f(-3) = -81 \sum_{h=0}^{\infty} \frac{(-1)^h}{2h+1}$$: convergeert

$S = [-3, 3]$

gesloten interval $\Rightarrow$ unif conti. (7-27)

4 a $f(x,y) = \begin{cases} (x-3y)^2 \arctan\left(\frac{1}{x-3y}\right) & x \neq 3y \\ 0 & x = 3y \end{cases}$

$$|f(x,y) - f(3y,y)| = \left|(x-3y)^2 \arctan\left(\frac{1}{x-3y}\right)\right|$$
$$\leq \frac{\pi}{2}(x-3y)^2 < \varepsilon$$

Stel $\delta = |(x,y) - (3y,y)| = |x-3y|$

dus $|f(x,y) - f(3y,y)| \leq \frac{\pi}{2}\delta^2 = \varepsilon$

f continu voor $x = 3y$, want voor geg. $\varepsilon > 0$, kies $\delta \leq \sqrt{\frac{2\varepsilon}{\pi}}$

b -

c $\frac{\partial f}{\partial x}(3y,y) = \lim_{h \to 0} \frac{f(3y+h,y) - f(3y,y)}{h}$

$$= \lim_{h \to 0} \frac{h^2 \arctan\frac{1}{h} - 0}{h} = 0.$$ insluitst.

Zelfde voor y.

d $\lim_{(h,h) \to (0,0)} \frac{f(h,h) - f(0,0) - \nabla f(0,0) \cdot (h,h)}{\|(h,h)\|}$

$$= \lim \frac{(h-3h)^2}{\sqrt{h^2+h^2}} \arctan\frac{1}{(h-3h)} \leq \lim \frac{\pi}{2} \frac{(h-3h)^2}{\sqrt{h^2+h^2}}$$

$= h^2 - 6hh + 9h^2$
$\leq h^2 + 3(h^2+h^2) + 9h^2$
$\leq 12(h^2+h^2)$

$$\leq \lim 6\pi\sqrt{h^2+h^2} = 0.$$

5 a zie boek

b bereken $\frac{\partial F_i}{\partial x_i}$.

$$\underline{F} = (u^2 + sx + ty,\ v^2 + tx + sy,\ 2s^2x + 2t^2y - 1,\ s^2x - t^2y)$$

$\underline{F}$ is C^1

$$\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(u, v, s, t)} = \det \begin{vmatrix} 2u & 0 & x & y \\ 0 & 2v & y & x \\ 0 & 0 & 4sx & 4ty \\ 0 & 0 & 2sx & -2ty \end{vmatrix}$$

$$= -69\, uvstyx \neq 0, \text{ want } u,v,s,t,x,y \neq 0$$

Implicitiete functiestelling zegt ja.

6 $\omega = x\,dy\,dz + y\,dz\,dx + z^2\,dx\,dy$ $\qquad \varphi(x,y,z) = (x, y, \sqrt{4x^2+y^2})$

$S:\ z = \sqrt{4x^2+y^2} \quad 0 \leq z \leq 1$

$$\iint_S \omega = \iint_S x\,dy\,dz + y\,dz\,dx + z^2\,dx\,dy$$

$$= \iint_S (x, y, z^2) \cdot \underline{N}_\varphi\, dx\,dy$$

$$N_\varphi = \varphi_x \times \varphi_y = \left(\frac{-4x}{\sqrt{4x^2+y^2}}, \frac{-y}{\sqrt{4x^2+y^2}}, 1 \right)$$

normaalvector van de z-as af: gaat goed.

$$\iint_S \omega = \iint_{4x^2+y^2 \leq 1} (x, y, z^2) \cdot N_\varphi\, dx\,dy$$

$$= -\iint_{4x^2+y^2 \leq 1} \left(\sqrt{4x^2+y^2} - 4x^2 + y^2 \right) dx\,dy$$

$$= -\int_{\vartheta=0}^{2\pi} \int_{R=0}^{1} (R - R^2)\, R\, dR\, d\vartheta$$

$x = \frac{R}{2}\cos\vartheta$

$y = R \sin\vartheta$

$R^2 = 4x^2 + y^2$

$$= -\frac{\pi}{3}$$