Introduction to Investment Theory (191515603), 2012-2013
Solution to Final Exam, 7-Nov-2012

1. (a) Let $s_{0.5}$ and s_1 denote the spot rates for 6 months and 12 months, respectively. Since compounding happens semi-annually, we have

$$\frac{10\,000}{\left(1+\frac{1}{2}s_1\right)^2} = 9\,564.74 \quad \text{and} \quad \frac{100\,000}{\left(1+\frac{1}{2}s_{0.5}\right)^1} = 98\,280.10.$$

It then follows that $s_{0.5} = 3.500\%$ and $s_1 = 4.500\%$.

(b) Let f be the (yearly) forward rate for the period between 6 months from now and 1 year from now. Then, we must have

$$\left(1+\frac{1}{2}s_{0.5}\right)\left(1+\frac{1}{2}f\right) = \left(1+\frac{1}{2}s_1\right)^2 \quad \Rightarrow \quad f = 2\left(\frac{\left(1+\frac{0.045}{2}\right)^2}{1+\frac{0.035}{2}} - 1\right) = 5.505\%.$$

Since the "theoretical" rate is higher than the offered rate of 5.45%, for investing money at this rate is definitely not a bargain.

(c) Contract F promises the delivery price, in 6 months, of bond A (1-year zero-coupon bond with face value 10 000) to be p_F. If expectations hypothesis has to be maintained then the current value of p_F (in 6 months time) must be same as the current price of bond A. In other words,

$$\frac{p_F}{\left(1+\frac{1}{2}s_{0.5}\right)} = \frac{10\,000}{\left(1+\frac{1}{2}s_1\right)^2} = 9\,564.74.$$

We have then $p_F = 9732.12$.

(d) When immunization is important one should look at the "duration" (not the maturity) of the relevant assets and the duration D_{portf} of the to be immunized portfolio must be between the lowest and highest duration of the available assets. For a zero-coupon bond duration is same as the maturity. Here the obligation (to pay 2 000 000 dollars in exactly 2 years) is like a zero-coupon bond (with only one payment at maturity). Hence the duration D_{oblg} is 2 years. But the available assets have duration 6 months and 1 year, both of which is smaller than 2 years. Hence, intuitively, it is impossible to create something secure for two years from now by using instruments which all expire within one year. We do not know what is going to happen after the expiry of the available assets (i.e., between one year from now and two years from now) and at this moment we do not have any instrument to protect us against that (unknown).

(e) The quasi-modified duration of the portfolio consisting of 2 bonds A and 2 bonds B can for example be given by

$$D_{qm} = \frac{0.5\cdot(2\cdot 10\,000)\left(1+\frac{1}{2}s_{0.5}\right)^{-2} + 1.0\cdot(2\cdot 100\,000)\left(1+\frac{1}{2}s_1\right)^{-3}}{(2\cdot 10\,000)\left(1+\frac{1}{2}s_{0.5}\right)^{-1} + (2\cdot 100\,000)\left(1+\frac{1}{2}s_1\right)^{-2}}.$$

(a) Let Z be a portfolio with capital/investment percentage of α in X and $(1-\alpha)$ in Y. Then the rate of return of portfolio Z is given by $r_{\mathrm{Z}} = \alpha\, r_{\mathrm{X}} + (1-\alpha)\, r_{\mathrm{Y}}$ with variance

$$\begin{aligned}
\sigma_{\mathrm{Z}}^2 &= \alpha^2\,\sigma_{\mathrm{X}}^2 + (1-\alpha)^2\,\sigma_{\mathrm{Y}}^2 + 2\,\alpha(1-\alpha)\,\mathrm{Cov(X,Y)}^2 \\
&= \alpha^2\,\sigma_{\mathrm{X}}^2 + (1-\alpha)^2\,\sigma_{\mathrm{Y}}^2 + 2\,\alpha(1-\alpha)\,\rho_{\mathrm{XY}}\,\sigma_{\mathrm{X}}\,\sigma_{\mathrm{Y}} \\
&= \alpha^2\,0.05^2 + (1-\alpha)^2\,0.1^2 + 2\,\alpha(1-\alpha)\,\frac{1}{4}\cdot 0.05\cdot 0.1 \\
&= 0.0025\,\alpha^2 + 0.01\,(1-\alpha)^2 + 0.0025\,\alpha(1-\alpha) \\
&= 0.0025\,\alpha + 0.01\,(1-\alpha)^2.
\end{aligned}$$

To minimize the variance with respect to α we set the derivative to zero:

$$0.0025 - 0.02\,(1-\alpha) = 0 \quad \Rightarrow \quad \alpha = \frac{0.02 - 0.0025}{0.02} = 0.875,$$

and check that the second derivative is positive, which is true because $0.02\,\alpha > 0$.

So the minimum variance portfolio is with 0.875 (i.e., $\frac{7}{8}$) fraction of wealth in X and the rest ($\frac{1}{8}$) in Y.

(b) For the "market portfolio" M it is given that $\sigma_{\mathrm{M}} = 0.08$. Furthermore, it is given that $\beta_{\mathrm{X}} = \frac{1}{2}$. Then the correlation coefficient between the rates of return of X and M is given by

$$\rho_{\mathrm{XM}} = \frac{\mathrm{Cov(X,M)}}{\sigma_{\mathrm{X}}\sigma_{\mathrm{M}}} = \frac{\mathrm{Cov(X,M)}}{\sigma_{\mathrm{M}}^2}\,\frac{\sigma_{\mathrm{M}}}{\sigma_{\mathrm{X}}} = \beta_{\mathrm{X}}\,\frac{\sigma_{\mathrm{M}}}{\sigma_{\mathrm{X}}} = \frac{1}{2}\cdot\frac{0.08}{0.05} = 0.8.$$

(c) The CAPM relationship $\bar{r}_{\mathrm{X}} - r_f = \beta_{\mathrm{X}}\,(\bar{r}_{\mathrm{M}} - r_f)$ leads to the relationship about the returns

$$r_{\mathrm{X}} = \text{constant} + \beta_{\mathrm{X}}\, r_{\mathrm{M}} + \epsilon$$

where ϵ denotes the firm-specific fluctuations in return (and uncorrelated with the market return). This leads to

$$\sigma_{\mathrm{X}}^2 = \beta_{\mathrm{X}}^2\,\sigma_{\mathrm{M}}^2 + \sigma_\epsilon^2.$$

Hence the firm-specific variance of the rate of return of X, expressed as percentage of the total variance is given by

$$\frac{\sigma_\epsilon^2}{\sigma_{\mathrm{X}}^2} = \frac{\sigma_{\mathrm{X}}^2 - \beta_{\mathrm{X}}^2\,\sigma_{\mathrm{M}}^2}{\sigma_{\mathrm{X}}^2} = \frac{0.05^2 - \left(\frac{1}{2}\right)^2\cdot 0.08^2}{0.05^2} = \frac{0.0009}{0.0025} = 0.36 = 36\%.$$

2. (a) Here all the contracts evolve around the three basic events/outcomes of the match:

(1) FC Twente wins (2) Feyenoord wins (3) Draw (nobody wins)

Let us denote the corresponding state prices by $\psi_1,\ \psi_2,\ \psi_3$. It then follows from the description that

$$1 = 1.67\,\psi_1, \qquad 1 = 4.50\,\psi_2, \qquad 1 = 3.60\,\psi_3$$

$$\Rightarrow \quad \psi_1 = \tfrac{1}{1.67} = 0.5988, \qquad \psi_2 = \tfrac{1}{4.50} = 0.2222, \qquad \psi_3 = \tfrac{1}{3.60} = 0.2778.$$

(b) The price of a (risk-free) contract, i.e., which will give you 1 in every possible situation, is given by $1\cdot\psi_1 + 1\cdot\psi_2 + 1\cdot\psi_3 = 1.0988$. Hence the rate of return for this risk-free investment is

$$r_f = \frac{\text{payoff} - \text{investment}}{\text{investment}} = \frac{1 - 1.0988}{1.0988} = -0.0899 = -8.99\%.$$

(c) The explanation behind negative risk-free rate is the profit of the bookmaker.

["bwin.com" sets its prices always keeping some profit margin in mind, whereas the standard theory we have learned in class/from book is always about "fair" price!]

(d) The risk-neutral probabilities are calculated by $q_i = \frac{\psi_i}{\psi_1+\psi_2+\psi_3}$. Hence, the risk-neutral probability that FC Twente will win the game is

$$q_1 = \frac{0.5988}{1.0988} = 0.5450 = 54.5\%.$$

(e) Suppose the price of the given contract is p. Then the payoff of "Twente Light" is

$$5, \quad \text{if Twente wins}, \qquad 0, \quad \text{if Twente loses}, \qquad p, \quad \text{if a draw}.$$

Hence we must have

$$p = 5 \cdot \psi_1 + 0 \cdot \psi_2 + p \cdot \psi_3 \quad \Rightarrow \quad p = \frac{5 \cdot \psi_1}{1 - \psi_3} = \frac{5 \times 0.5988}{1 - 0.2778} = 4.15.$$

(f) Supposed x is the amount of bet on FC Twente. Then the wealth after the game is

$$X = \begin{cases} 100 + 0.67\,x & \text{if FC Twente wins} \quad \text{i.e., with probability } 0.7 \\ 100 - x & \text{otherwise} \quad \text{i.e., with probability } 0.3 \end{cases}$$

With logarithmic utility $U(\cdot) = \ln(\cdot)$, the expected utility of the wealth after the bet is

$$E[U(X)] = 0.7 \cdot \ln(100 + 0.67\,x) + 0.3 \cdot \ln(100 - x).$$

To maximize the expected utility we set the derivative with respect to x to zero:

$$\frac{0.7 \times 0.67}{100 + 0.67\,x} - \frac{0.3}{100 - x} = 0 \quad \Rightarrow \quad 46.9 - 0.469\,x = 30 + 0.201\,x$$

$$\Rightarrow \quad x = \frac{46.9 - 30}{0.469 + 0.201} = 25.22,$$

and check that the second derivative is negative, which is true because

$$-\frac{0.7 \times 0.67^2}{(100 + 0.67\,x)^2} - \frac{0.3}{(100 - x)^2} < 0.$$

Hence the bet should be 25.22 euro.

3. Answer should touch upon ....

Mean variance analysis summarizes all the randomness by two quantities : mean (i.e., the expected return : more the better) and variance (i.e., variability in the returns : less the better). A portfolio can be analyzed by making a balance between this two quantities, for example by Sharpe index = mean/standard deviation.

Markowitz portfolio model uses the mean-variance approach to set up an optimization problem to create a portfolio with minimum variance but with a given mean rate of return. A portfolio can be analyzed using this method to see whether another set/combination of assets can bring for example same return but with less risk (= variability in the returns).

CAPM theory is an "equilibrium theory". According to CAPM the expected rate of return from an "efficient" portfolio (i.e., one which cannot be improved on both fronts – return and risk) can be determined completely from the market rate of return and risk-free return (and the beta coefficient of the portfolio) : $r = r_f + \beta(r_M - r_f)$. A portfolio can be analyzed by looking at its beta-coefficient (the sensitivity with the market) and evaluating whether it is at the desired level. Furthermore, one can also look at the past history (empirical study) to see how the portfolio returns fared as compared to the CAPM predictions (as measured by Jensen's index).