Exam march 20, 2000

1) use hours. 6 per day $\rightarrow$ $\frac{1}{4}$ per hour $\lambda = \frac{1}{4}$

$N(t)$ = # arrivals in interval of length t $\sim$ Poisson(λt)

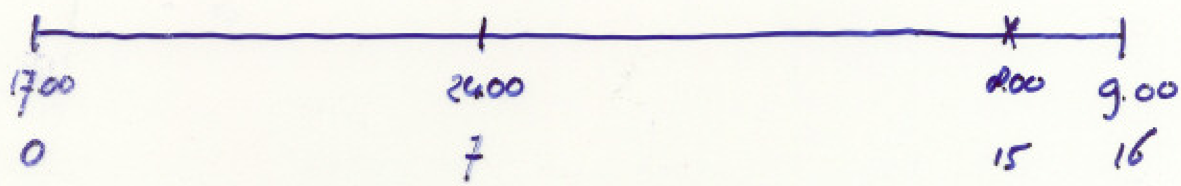
X = typical interarrival time $\sim \exp(\lambda)$

a) $P(N(16)=0) = e^{-\lambda \cdot 16} \cdot \frac{(16\lambda)^0}{0!} = e^{-4} \approx 0{,}018$

or: $P(X > 16) = e^{-\lambda \cdot 16} = e^{-4}$

b) $P(N(48)=2) = e^{-\lambda \cdot 48} \cdot \frac{(48\lambda)^2}{2!} = 72\, e^{-12} \approx 0{,}000442$

c) Choose $t=0$ at 17.00 hr. the previous day:



Let $N(t)$ = # arrivals in $(0,t]$

$$P(N(7)=0 \mid N(15)=2) = \frac{P(N(7)=0,\ N(15)=2)}{P(N(15)=2)}$$

$$= \frac{P(N(7)=0,\ N(15)-N(7)=2)}{P(N(15)=2)} = \frac{e^{-7\lambda}\frac{(7\lambda)^0}{0!} \cdot e^{-8\lambda}\frac{(8\lambda)^2}{2!}}{e^{-15\lambda}\frac{(15\lambda)^2}{2!}} = \frac{8^2}{15^2} = \frac{64}{225}$$

or use binomial distr.

d) Time until next arrival =

remaining interarrival time, also $\sim \exp(\lambda)$

Expectation $= \frac{1}{\lambda} = 4$ hours

$\Rightarrow$ Expected time of next arrival is 13.00 hour

3) Discrete time Markov chain (X_n, Y_n) with X_n (Y_n) = # components of type 1 (2) after n hours.

States: $(2,0)$ $(1,0)$ $(0,0)$ $(0,1)$ $(0,2)$

or use $Z_n = Y_n - X_n$: -2 -1 0 1 2

- If $Z_n = -2$, machine 1 does not produce, and

$$P(Z_{n+1} = -1 \mid Z_n = -2) = P(\text{component 2 good}) = \frac{3}{4}$$
$$P(Z_{n+1} = -2 \mid Z_n = -2) = P(\text{component 2 bad}) = \frac{1}{4}$$

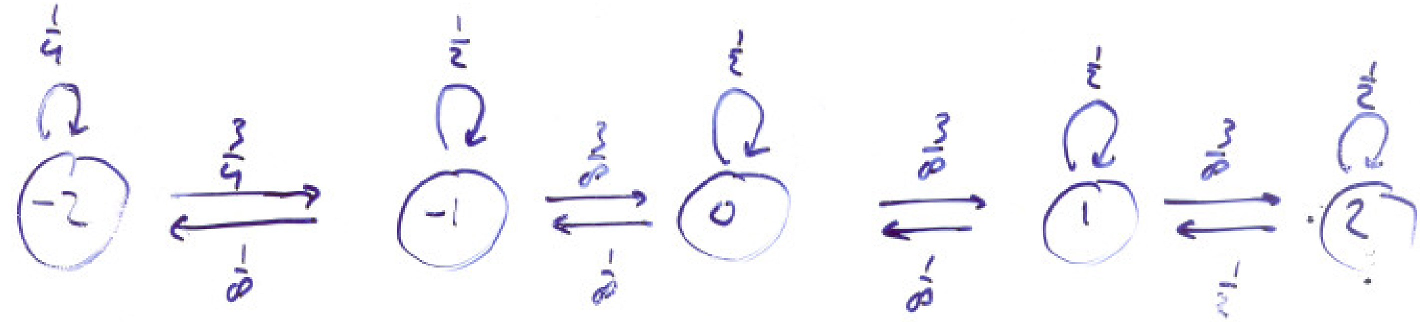
- If $Z_n = 2$, machine 2 does not produce, and

$$P(Z_{n+1} = 1 \mid Z_n = 2) = P(\text{component 1 good}) = \frac{1}{2}$$
$$P(Z_{n+1} = 2 \mid Z_n = 2) = P(\text{component 1 bad}) = \frac{1}{2}$$

- If $Z_n = i$, $i = -1, 0, 1$, both machines produce and

$$P(Z_{n+1} = i-1 \mid Z_n = i) = P(\text{1 good, 2 bad}) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$
$$P(Z_{n+1} = i+1 \mid Z_n = i) = P(\text{1 bad, 2 good}) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$
$$P(Z_{n+1} = i \mid Z_n = i) = P(\text{both good}) + P(\text{both bad})$$
$$= \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2} \qquad \left(= 1 - \frac{1}{8} - \frac{3}{8}\right)$$

Transition diagram:



3a) Long run fraction of time machine 1 is not working is $\pi_{-2} = \lim_{n\to\infty} P(Z_n = -2)$

For machine 2: $\pi_2 = \lim_{n\to\infty} P(Z_n = 2)$

Solve $\begin{cases} \underline{\pi} = \underline{\pi} P \\ \sum_{i=-2}^{2} \pi_i = 1 \end{cases}$ with $P = \frac{1}{8}\begin{pmatrix} 2 & 6 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix}$

$\Rightarrow \underline{\pi} = \frac{1}{239}(2, 12, 36, 108, 81)$

$\pi_{-2} = \frac{2}{239}$ and $\pi_2 = \frac{81}{239}$

b) Expected # products per hour (production per hour: 0 or 1)

$= \lim_{n\to\infty} P(\text{product in period } n)$

$= \lim_{n\to\infty} \sum_{i=-2}^{2} P(\text{product in period } n \mid Z_n = i) \cdot P(Z_n = i)$

$= \frac{3}{4}\pi_{-2} + \frac{3}{4}\pi_{-1} + \frac{3}{8}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 = \frac{237}{478}$

or: each type-1-component eventually leads to a product, so

$\underbrace{(1-\pi_{-2})}_{\text{fraction of time machine 1 is working}} \cdot \underbrace{\tfrac{1}{2}}_{P(\text{component 1 good})} = \frac{237}{478}$

or: $(1-\pi_2)\cdot\frac{3}{4} = \frac{237}{478}$

4a) $X(t)$ = # available machines (not broken down)

state space: $\{0, 1, 2, 3\}$

lifetimes $\sim \exp\left(\frac{1}{4}\right)$

repair times $\sim \exp\left(\frac{1}{2}\right)$

Note: $\{X(t)\}$ must be a birth and death process, since only 1 event occurs at a time.

In state 1: 1 machine active, 2 in repair.

time until next repair $\sim \exp(1)$ $\quad 2 \cdot \frac{1}{2}$

" " " breakdown $\sim \exp\left(\frac{1}{4}\right)$

$\Rightarrow$ sojourn time in 1 is $\sim \exp\left(\frac{5}{4}\right)$ $\qquad v_1 = \frac{5}{4}$

$q_{10} = \frac{5}{4} \cdot \frac{1/4}{1 + 1/4} = \frac{1}{4}$ $\qquad$ ($= \mu_1$, death rate in state 1)

$q_{12} = \frac{5}{4} \cdot \frac{1}{1 + 1/4} = 1$ $\qquad$ ($= \lambda_1$, birth rate in state 1)

P_{12}

and so on:

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Note: time until first repair in 0 is $\sim \exp\left(\underline{\underline{2}} \cdot \frac{1}{2}\right)$ (2 repairmen)

" " " breakdown in 3 is $\sim \exp\left(\underline{\underline{2}} \cdot \frac{1}{4}\right)$

(2 machines active)

4b) Solve $\begin{cases} \underline{\pi}\ Q = \underline{0} \\ \sum_{i=0}^{3} \pi_i = 1 \end{cases}$ $\Rightarrow \underline{\pi} = \frac{1}{21}(1, 4, 8, 8)$

or $\begin{cases} \lambda_i \pi_i = \mu_{i+1} \pi_{i+1}, \quad i = 0,1,2 \\ \sum \pi_i = 1 \end{cases}$

c) Expected long run production per hour

$= 700 \cdot$ Expected long run # active machines

$= 700 \cdot (0\,\pi_0 + 1\,\pi_1 + 2\,\pi_2 + \underline{\underline{2}}\,\pi_3)$

$= 700 \cdot \frac{12}{7} = 1200$

a) alternating renewal process (between 0 and $\{1,2,3\}$)

$$\lim_{t\to\infty} P(X(t)=0) = \frac{E(\text{length of sojourn in } 0)}{E(\text{cycle length})}$$

$$\Leftrightarrow \quad \pi_0 = \frac{-1/q_{00}}{-1/q_{00} + E(\text{length of sojourn in } \{1,2,3\})}$$

$$\Rightarrow \frac{1}{21} = \frac{1}{1 + E(\ldots)} \quad \Rightarrow E(\ldots) = 20$$

or using that $\{X(t)\}$ is a Markov chain:

d) Make state 0 absorbing

Let $m_i = E(\text{time to absorption} \mid X(0)=i)$. Find $\underline{m_1}$

$$\left.\begin{aligned} m_1 &= \tfrac{4}{5} + \tfrac{4}{5} m_2 \\ m_2 &= 1 + \tfrac{1}{2} m_1 + \tfrac{1}{2} m_3 \\ m_3 &= 2 + m_2 \end{aligned}\right\} \Rightarrow \underline{m_1 = 20} \quad m_2 = 24 \quad m_3 = 26$$