Complex Function Theory
(152025)

Thursday, 2009, June 25, 9-12am

- To every answer, a motivation is required.
- You may use a (graphic) calculator.
- For those who have gained the homework bonus, they do not need to solve exercise 6, and still get these four points (see the score table below).

1. (a) Find all $z \in \mathbb{C}$ for which $\sin(z) = \cos(z)$. 3

   (b) For which $z \in \mathbb{C}$ is $Log(i^z) = z\, Log(i)$? 3 1

2. The complex function $f(z)$ is analytic for all $z \in \mathbb{C}$.
   Denote $z = x + iy$, $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f) = v$.

   (a) Formulate the Cauchy-Riemann relations between u and v and express $f'(z)$ in terms of partial derivatives of u and v. 2

   (b) For $f(z)$ it is known that $\frac{\partial v}{\partial x} = 12xy - 6x$ and $f(0) = 3 - 2i$ and $f'(0) = 1$. Find the function $f(z)$.
   (Hint: determine first the function $f'(z)$). 3 4

3. Compute

$$\int_\Gamma \frac{\cos(z)}{z^2(z-3)} dz$$

5 3

   along the contour $\Gamma : |z| = 4$.

4. Find the Laurent series for the function

$$\frac{z}{(z+1)(z-2)}$$

   in each of the following domains:

   (a) $|z| < 1$ 2

   (b) $2 < |z|$. 2 4

5. Verify: 6 4

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{3x}} dx = \frac{2\pi}{3\sqrt{3}}$$

(Hint: use the residue theorem for a rectangle based on the real axis with height $\frac{2\pi}{3}$).

6. (a) Formulate Rouche's Theorem for analytical functions.

(b) Prove, using Rouche's Theorem, that every polynomial of degree n has n zeros. 3 4

7. Verify that the Fourier transform of

$$F(t) = \frac{1}{t^2 + a^2}$$

is for $a > 0$ given by 6 4

$$G(\omega) = \frac{e^{-a|\omega|}}{2a} \qquad (\omega \in \mathbb{R})$$

Grading points:

1.(a) 3 (b) 3	2. (a) 2 (b) 3	3. 5	4. (a) 2 (b) 2	5. 6	6. (a) 1 (b) 3	7. 6

Total: 36+4 = 40 points