Exam Complex Function theory
Code 2018-201500405

Date : Tuesday 28 May 2019
Place : ??
Time : 08.45 – 10.45

All answers must be motivated.
The use of a pocket calculator is not allowed.

1. Define on $\Omega := \{z \in \mathbb{C} \mid z \neq 0\}$ the function

$$f(z) = \frac{e^z - 1}{z}, \quad z \in \Omega. \tag{1}$$

   (a) Prove that f is analytic on Ω.

   (b) Show that f can be extended to an entire function. Prove that this extension is unique.

2. Given is the function

$$v(x, y) = x^3 - 3xy^2 + y^2 - x^2 - x. \tag{2}$$

   (a) Show that v is harmonic on $\mathbb{R}^2$.

   (b) Suppose that $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, with f entire and v given by (2). Find $f(z)$, as function of z, when additionally it is given that $f(0) = 0$.

3. (a) Find the largest domain on which $\mathrm{Log}(z^2)$ is analytic.

   (b) Give an example of two complex numbers z_1 and z_2 such that the principle value of $(z_1 z_2)^{\frac{1}{2}}$ is not equal to the product of the principle values $z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}$.

4. Determine the integral of $\cos(z)$ along the curve $\gamma = \{\frac{e^{2it}}{e^{it}-2} \mid t \in [0, \pi]\}$, see Figure 1

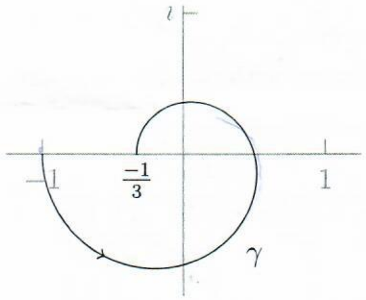

Figure 1: The curve γ.

P.T.O.

5. Let f be an entire function. Assume that for all real x there holds $f(x) = f(-x)$. Prove that $f(z) = f(-z)$ for all $z \in \mathbb{C}$.

6. Let f be a meromorphic function. Assume that 0 is a pole of f and that it is the only pole within the closed unit circle $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$. Show that f is unbounded in the right unit disc. Thus

$$\sup_{z \in \mathbb{D}, \mathrm{Re}(z) > 0} |f(z)| = \infty.$$

7. Consider the function

$$g(z) = e^{-1/z}, \quad z \neq 0$$

   (a) Show that g is bounded on $\{z \in \mathbb{C} \mid |z| \leq 1, \mathrm{Re}(z) > 0\}$ and unbounded on $\{z \in \mathbb{C} \mid |z| \leq 1, \mathrm{Re}(z) < 0\}$.

   (b) Explain why the above is not in contradiction with the result of Exercise 6.

8. Use Principle of the Argument to determine the integral

$$\int_C \frac{z \cos(z^2)}{\sin(z^2)} dz.$$

Here C is the unit circle, positively oriented.

9. Determine the integral

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 1} dx,$$

where P.V. stands for principle value.

10. Consider the function

$$h(z) = z + \pi + e^{-z}. \tag{3}$$

Let Ω_R^+ be the half disc $\Omega_R^+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, |z| \leq R\}$, with $R > 0$.

   (a) Show that $h(z)$ has no zeros in Ω_1^+.

   (b) Show that $h(z)$ has no zeros with positive real part.

Points[1]

Ex. 1		Ex. 2		Ex. 3		Ex. 4	Ex. 5	Ex. 6	Ex. 7		Ex. 8	Ex. 9	Ex. 10	
a	1	a	1	a	2	2	3	3	a	2	4	4	a	2
b	3	b	4	b	2				b	1			b	2

[1]Total: $36 + 4 = 40$ points