Exam Complex Function theory
Code 2019-201500380

Date	:	Friday, July 10, 2020
Place	:	Home
Time	:	8.45 – 10.45

All answers must be motivated.
If you have questions, then you can use scat function of Canvas or phone: 053-489???
The use of the book is allowed. However, it is not allowed to let you be assisted or to help any other, see also the statement at the end of the exam.

1. Given the function
$$v(x,y) = 3x^2y - y^3 + 3y + 6 \tag{1}$$
   (a) Show that v is harmonic on $\mathbb{R}^2$.
   (b) Suppose that $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$, with f entire and v given by (1). Find $f(z)$, as function of z, when additionally it is given that $f(0) = i$.

2. The Fibonacci sequence $\{a_n, n \geq 0\}$, is given by $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$, $n \in \mathbb{N}$, $n \geq 2$. It can be shown that they can also be expressed as
$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right], \qquad n \geq 0. \tag{2}$$
   (a) Show that the power series $f(z) = a_0 + a_1 z + a_2 z^2 \cdots$ defines an analytic function and determine the circle of convergence.
   (b) Prove that f satisfies
   $$f(z) = 1 + zf(z) + z^2 f(z)$$
   and determine $f(z)$.

3. Determine whether the following assertions are true or false:
   (a) The following equalities all hold
   $$i^{4i} = \left(i^4\right)^i = 1^i = 1.$$
   (b) If f is analytic inside the domain D and γ is simple closed positively oriented contour inside D, then
   $$\int_\gamma \frac{f'(z)}{z - z_0} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz.$$
   Here $z_0 \in D$ but not on γ.

P.T.O.

4. (a) Determine for $R > 0$, $R \neq 1$, the integral $\int_{\Gamma_R} \frac{1}{z^3+1} dz$, where Γ_R is given in Figure 1.

   (b) Determine the integral $\int_0^\infty \frac{1}{x^3+1} dx$.

   Hint: if needed you may use that $e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $e^{2i\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

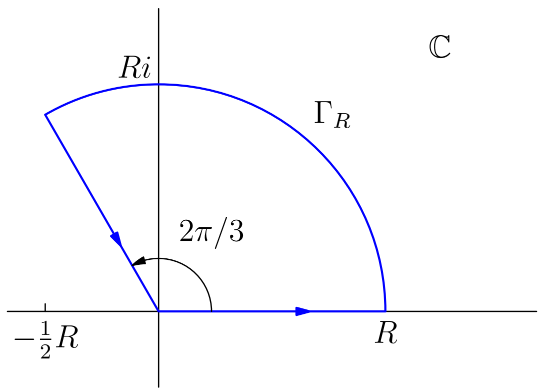

Figure 1: The contour Γ_R; a circular arc of radius R connected to the origin by 2 straight lines.

5. The outcome of the integral

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{e^z - 2i}{e^z - 2iz + 2 + i} dz, \tag{3}$$

   where Γ_R is drawn in Figure 1, is given in the following table:

R	(3)	R	(3)
1	0	6	1
2	0	7	1
3	0	8	1
4	1	9	1
5	1	10	2

   What can you conclude from this table about the zero's on $e^z - 2iz + 2 + i$?

6. Consider the function

$$h(z) = z^7 + z^2 + \frac{5}{2}. \tag{4}$$

   (a) Show that h has no zero's with modulus less than 1.

   (b) Show that h has seven zero's of modulus less than 2.

Points[1]

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[1] Total: $36 + 4 = 40$ points

Statement:

Copy the statement below on a separate sheet of paper, and hand it in together with your exam solutions.

I, `insert your name`, hereby declare that I have made this test to the best of my own ability. I personally guarantee that I have made this test without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

Date and place: . . . .