Exam Complex Function theory
Code 201500405

Date : Thursday June 3, 2021
Place : NH-205
Time : 13.45 – 15.45

All answers must be motivated.
The use of a pocket calculator or any other electronic equipment is not allowed.

1. Show, by using the definition, that $f(z)$ is analytic at the origin, where $f(z)$ is given by

$$f(z) = \begin{cases} \frac{\sin(z)}{z} & z \neq 0, \\ 1 & z = 0. \end{cases}$$

2. Given is the function

$$u(x, y) = -2xy + 2x + 5. \qquad (1)$$

   (a) Show that u is harmonic on $\mathbb{R}^2$.

   (b) Suppose that $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, with f entire and u given by (1). Find $f(z)$, as function of z, when additionally it is given that $f(0) = 1$.

3. Let α be a complex number. Show that for $z \neq 0$ the following holds,

$$z^{-\alpha} = \frac{1}{z^{\alpha}},$$

   where each power function is given by its principal branch.

4. Let Γ denote the unit circle, i.e., $\Gamma = \{z \in \mathbb{C} \mid |z| = 1\}$. Does there exists a meromorphic function f with zero being a pole of f, such that the following holds:

$$\oint_{\Gamma} f(z)dz = 0.$$

   If yes, construct such a function. If not, prove why.

5. Picard theorem states that a function with an essential singularity assumes every complex number, with possible one exception, as a value in any neighbourhood of this singularity.

   Verify Picard's theorem for $e^{1/z}$ near $z = 0$.

P.T.O.

6. Determine the outcome of the following integral

$$\int_\Gamma \frac{z^2}{z^2-1}\,dz,$$

where Γ is the contour drawn in Figure 1

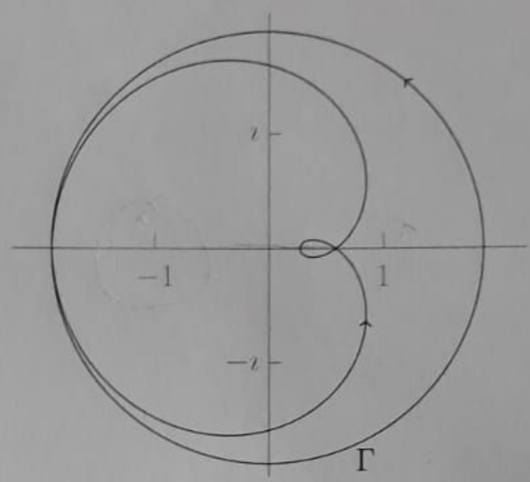

Figure 1: The curve Γ.

7. Let $g(z)$ be an entire function, and define $z_n = n^{-1}$ for $n \in \mathbb{Z}$. Show that if $g(z_n) = 0$ for all $n \in \mathbb{Z}$, then g is identically zero.

8. Determine the integral

$$\text{P.V.} \int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)}\,dx,$$

$\frac{1}{2}\cdot\frac{1}{4}\pi = \frac{1}{8}\pi$

where P.V. stands for principle value.

9. Suppose that $f(z)$ is analytic on a domain containing the closed unit disc, $\{z \in \mathbb{C} \mid |z| \leq 1\}$, and satisfies $|f(z)| < 1$ for $|z| = 1$.

   Show that the equation $f(z) = z$ has exactly one (counting multiplicity) solution inside the open unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$.

Points[1]

Ex. 1	Ex. 2		Ex. 3	Ex. 4	Ex. 5	Ex. 6	Ex. 7	Ex. 8	Ex. 9
3	a	2	3	4	3	5	4	4	4
	b	4							

[1] Total: 36 + 4 = 40 points